BEHAVIOR OF IDEAL GEODESICS ON HYPERBOLIC 2- MANIFOLDS

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Abstract
In this paper we review some important results on the behavior of ideal geodesics problem for some hyperbolic 2-manifolds with cusps, and discuss some extension of those results to the case of a hyperbolic surface with genus g and k punctures.

***Key words** : behavior of geodesics, ideal geodesics, hyperbolic surface with genus g and k puncture, Këbe method, the method of colour multilaterals*

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1. INTRODUCTION

This paper focuses on the problem of classification of geodesics (behavior) on hyperbolic 2 dimensional manifolds. One of the most important problems in geometry is to find the shortest path or geodesic between the two points. Behavior of geodesics in hyperbolic surfaces has been a fruitful subject of research for many years. Such geodesics are often studied by looking at their lifts in covering spaces of the surface. In section 2 we compute classification of bicuspidal and unicuspidal geodesics on the simplest surfaces. In section 3 we explore a method of colour multilaterals for investigating the behavior of ideal geodesics on hyperbolic 2 dimensional manifolds with cusps.

The following terminology will be used regularly throughout this paper. Let M be a finite area hyperbolic surface. Then M is homeomorphic to a closed surface with finitely many points removed. Each of these points called punctures , has special neighborhoods in M called cusp. A geodesic in a hyperbolic manifold is a locally distance - minimising curve, and is said to be simple if it has no transverse self-intersections (there-fore it is either an embedded copy of R or an embedded circle) and non simple otherwise. A geodesic on surface M is said to be complete if it is not strictly contained in any other geodesic, i.e., it is either closed and smooth, or open and of infinite length in both directions. Complete geodesics coincide with those which never intersect ∂M . Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp. Throughout, we use the term geodesic to refer to a complete infinite geodesic; a geodesic ray is a half-infinite ray; finally, a geodesic arc is a finite segment lying along some geodesic (which we assume to be closed unless otherwise stated).

2. BEHAVIOR OF BICUSPIDAL AND UNICUSPIDAL GEODESICS ON THE SIMPLEST 2 DIMENSIONAL MANIFOLDS

We investigate in detail the global behavior of the ideal geodesics on the simplest hyperbolic surfaces: hyperbolic horn (funnel end), hyperbolic cylinder and parabolic horn (cusp, horn end), or parabolic cylinder [1]. A hyperbolic horn is a two-dimensional manifold , obtained from the strip between the two parallel straight lines of the hyperbolic plane by matching the border lines by shifting (sliding), its axis being parallel to the border lines and beyond the strip between them. The hyperbolic horn, i.e. the factor-space $H^2_+\backslash\Gamma$, is an (open) half of the hyperbolic cylinder. The border circumference does not belong to that half and there for the surface of the hyperbolic horn is incomplete. The funnel is half of the hyperbolic cylinder, bounded by their closed geodesic. The full funnel continues to flare out exponentially and has infinite area. So, every geodesic curve γ on the hyperbolic horn is of one of the four types: a) there are no closed geodesics; b) there is a geodesic of infinite length, without self-intersections points, and any of its points divides the geodesic into two rays: one ray of finite length and another ray of infinite length; c) there is an infinite geodesic, without self-intersections points and any of its points divides it into two congruent rays; d) there is an infinite geodesic and it has a finite number *k* of double self-intersection points and they are all divisible by 2. The number *k* of self-intersection points of an examined geodesic is equal to *p.* The problem of behavior of a geodesic on a hyperbolic cylinder is solvable. One may define the hyperbolic cylinder as a non-compact two-dimensional manifold obtained from the strip from between the two divergent lines of the hyperbolic plane by identifying the divergent border lines by shift (sliding), its axis being a common perpendicular for the said border lines, its shift being equal to the length of such translation. The factor space $H^2\backslash\Gamma$ is a some kind of cylindrical surface also called hyperbolic cylinder. The hyperbolic cylinder is the union of two funnels. There are no closed geodesics on the cylinder *C* (both simple, different from the narrow geodesic core of cylinder and non-simple ones). There are a geodesic without self-intersection points, infinite in both directions (at both ends) on the cylinder. We shall call a parabolic horn (cusp) the two-dimensional manifold obtained from the strip from between the two parallel lines of the hyperbolic plane by identifying the border lines by horocyclic rotation determined by these lines. The parabolic cylinder is a special case (its small end is a cusp, while the "horn" end carriers the hyperbolic metric). The study of universal cover of parabolic cusp demonstrates that. So, every geodesic curve γ on the parabolic horn (cusp) is of one of the three types: *(*simple infinite length, without self-intersection; the geodesic is infinite in both directions (at both ends) and it has only a finite number k of double self-intersection points, in the particular case, both ends of the geodesic can go to the some point at infinity; there are no closed geodesics on parabolic cusp) [2].

3.BEHAVIOR OF IDEAL GEODESICS IN 2 DIMENSIONAL HYPERBOLIC MANIFOLDS WITH CUSPS

Let M be an orientable hyperbolic 2- manifold (surface) with genus *g* and *k* punctures and let γ be a geodesic in M. Every hyperbolic surface can be decomposed into "pairs of pants" (Y pieces). Given a surface of genus $g \geq 2$, there are $3g-3$ simple closed pairwise non-intersecting geodesics which partition the surface into $g-1$ such pieces. Every pairs of pants is constructed by pasting together tho copies of a right-angled geodesic hexagon. There exists a right-angled geodesic hexagon in the hyperbolic plane with pairwise non-adjacent sides of any prescribed lengths. The study of the geodesics on hyperbolic surfaces can be reduced to the study of the curves on a hyperbolic pair of pants. Compact hyperbolic surfaces can be seen as an elementary pasting of geodesic polygons of the hyperbolic plane. Conversely, cutting such a surface along disjoint simple closed geodesics (a partition), one obtains a family of pair of pants (surfaces of signature (0,3)), which in turn can be readily cut to obtain a pair of isometric right-angled hexagons. Let *M* be a surface and let *P* be a pair of pants. In this paper, we focus on getting the behavior of geodesics on a hyperbolic pair of pants *P*. As a direct consequence we get the behavior of geodesics on any surface *M*. We do this as follows. First, there is a unique way to write *P* as the union of two congruent right-angled hexagons. Take this decomposition.We examining different types of behaviors exhibited by geodesics on a given pair of hyperbolic pants and study infinite simple geodesic rays and complete geodesics. The construction can be extended to the situation when both remaining boundary components of the pair of pants are represented by cusps. We also allow the degenerate case in which one or more of the lenths vanish (a generalized pair of pants). For the behaviour of the geodesics on the specified fragments (hyperbolic pants, etc.) it is used a certain figure, named in the text of the work *the multilateral.* The sides of this „ *multilateral*" (without vertices and angles) are straight lines, tangents (regular) system of circumferences on the hyperbolic plane H^2. Obviously, the reflections in the sides of the straight lines of this multilateral can cover the whole (entire) hyperbolic plane H^2. To facilitate the understanding and further description, we agree to call the sides of the six-rectangle (right angled hexagon) black, if they are obtained from boundary geodesic circles of pants, and the other three sides we agree to consider painted in different colours (for example, red, blue and green straight). Exactly, this figure is also called in the work as a multilateral (in contrast to the polygon, the figure has no vertices and angles, hence its name - the multilateral). The study of the behavior of the geodesics in this paper is being carried out gradually, in order of collecting the surface, the reverse order of cutting the surface into fragments (i.e. pants). The surface is cut into typical pieces (for example, on pants or their degenerations, on right hexagons, etc.) and the question of the behavior of the geodesics for each piece is solved on it, and then the result of the investigation returns (by gluing) onto the original surface [3].

With the help of these multilaterals, it is possible to determine the nature of the behavior of the geodesics on the surface, not more complicated than how Artin studied the global behavior of geodesics on hyperbolic surfaces by cleverly encoding geodesics using continued fractions. Any given hyperbolic (closed, i.e., ordinary) surface can be cut into pants and the question is how, when gluing such pants, connect them on a common surface. But it may seem (when gluing of the surface from the pants is not finished yet) that the surface of genus g has also n components (the surface has a geodesic boundary). And, going further, we notice that the boundary of the surface can degenerate: transform into cuspidal ends (cusps) and into conical points. Thus, we arrive at the most general case, the surfaces of the signature (g, n, k), the preliminary investigation of the behaviour of the geodesics on these pieces. To summarize what has been said, we can conclude that a concrete method of investigating the behavior of the geodesics on hyperbolic 2-manifolds is based on the idea of preliminary research on these pieces (on the set of hyperbolic pants and their degenerations), in the subsequent consolidation of research results using the method proposed in this paper (sometimes called the method of generalized coloured multilaterals). In more detail, the following main results of the study were obtained. A new constructive method for investigating the global behaviour of the geodesics on hyperbolic manifolds (the method of colour multilaterals) is given in this paper [4]. The solution is based on the study of the behavior of the geodesics on the simplest hyperbolic surfaces (hyperbolic pants, degenerate hyperbolic pants, thrice-punctured sphere, etc.), some of which have long attracted the attention of geometers. In this paper is used the Këbe method of geodesic cutting of hyperbolic 2-manifolds into hyperbolic pants with a nonempty boundary (edge). In hyperbolic geometry, hyperbolic right angled hexagons are used as a tool for analysing the behaviour of the geodesic (and surfaces). The discrete group Γ is defined in the usual way via its fundamental domain F (glued from the proper number of right angled hexagons). Hyperbolic pants are the only compact hyperbolic surfaces with a geodesic boundary that can't be simplified by cutting along closed simple geodesics. In fact, any pants with boundary geodesics are uniquely determined by the length of their boundary geodesics, because any hyperbolic right angled hexagon is uniquely defined by three alternating (non-adjacent) lengths of sides that can be arbitrarily set. We consider the universal covering of hyperbolic pants (the hyperbolic plane H^2) and lines that cover a given geodesic. Let it be on H^2 a right angled hexagon H and let H' denote its image under reflection from the side of δ_{13}. When identifying the corresponding sides δ_{12} and δ_{12}, as well as δ_{23} and δ_{23} of this right angled geodesic octagon , we obtain hyperbolic pants P with boundaries $\alpha_1, \alpha_2, \alpha_3$. As the fundamental region for the corresponding pants P we choose this hyperbolic right angled octagon (the double of a right angled hexagon H in the plane H^2).This geodesic right angled octagon is the fundamental domain of the group $\Gamma = \langle t_1, t_2, t_3 \rangle$ generated by hyperbolic translations t_1, t_2, t_3, (the pants can be obtained by factorizing the hyperbolic plane H^2 by a discrete co-compact group Γ generated by translations t_1, t_2, t_3, where the translation t_i is determined by the vector $2 \cdot \alpha_i, i = 1,2,3$). To describe the behaviour of an arbitrarily given (some) geodesic on hyperbolic pants emanating from the point A in a given direction, we need to consider how the direct, covering this geodesic, behaves on the universal cover of these pants. In other words, how this straight line is located relative to the sides of $\alpha_3, \alpha_1, \alpha_2$ of the hyperbolic right angled octagon (the so-called "colour" straight - blue, green, red). Walking along hyperbolic octagon, we can't cross the boundary components $\alpha_3, \alpha_1, \alpha_2$ (" coloured circumferences "), but we can pass through the sides δ_{13}, δ_{12}, δ_{23} of a hyperbolic hexagon (the so-called "black" sides). Along with the coloured sides, the categories of coloured angles are built. A pair of "adjacent" colour angles uniquely determines the next colour angle with the help of colour (coloured "straight lines - blue, green, red) or with the help of geodesic sides $\alpha_1, \alpha_2, \alpha_3$. Suppose, for the beginning, point A is fixed on the surface of hyperbolic pants, and we need to understand how the geodesic's behaviour depends on the direction (from the directing vector emanating from point A). In this situation each side determines the angle of its colour with the vertex in the point A (and the sides parallel to the colour side) and in each category of angles it is uniquely determined which sides δ_{13}, δ_{12}, δ_{23} (or "black" sides) it is necessary to cross to be within the scope of the corresponding colour side. Thus, on hyperbolic pants, the problem of the behaviour of any geodesic passing through a fixed point is uniquely solvable by the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral obtained from a right angled hexagon. Thus, *on hyperbolic pants is the problem of the behaviour of any geodesic* that passes through a fixed point and is uniquely solvable with the help of *the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral*, obtained from a right angled hexagon. Further, the concept of the category of angles is introduced, and with the help of these categories an algorithm for recognizing the type of a geodesic is given.

Main results of the present work are as follows. In the work is given a new constructive method (a new approach) for solving the problem of the behavior of geodesic on a arbitrary hyperbolic surfaces of signature (g, n, k), i.e., method allowing to answer the question about the structure on the global of examined geodesic at its indefinitely extension (geodesics can be extended indefinitely) on both directions. Such a compressed formulated result can be disclosed as follows.

For this purpose, with the help of proposed practical approach at first are studied geodesics at the simplest hyperbolic manifolds : 1) it is solved the problem of the behavior of geodesic on the simplest hyperbolic surfaces (hyperbolic horn; hyperbolic cylinder; parabolic horn (cusp)); 2) it is investigated and described the behavior of the geodesic lines on hyperbolic surfaces of signature (0,3) (hyperbolic pants); it is found special case: behavior of ortho-boundary geodesics and orthogeodesics, 3) it is investigated and found behavior of the geodesics on compact closed hyperbolic surface without boundaries (borders), (general case). As specific problems are solved the following tasks: 4) there are studied geodesics on hyperbolic surface of genus g and n (non-puncture) boundary holes (geodesic boundaries); it is given characteristics of all possible types of geodesic launched orthogonally from the point of geodesic boundary of the surface, it is described their behavior and general structure; are studied intervals (horocyclic segments) formed by simple - normal geodesics, launched from the selected conical point, cusp or boundary geodesics on hyperbolic surface. 5) a) there are given the characteristics and there are studied properties and types of the geodesics on hyperbolic 1- punctured torus; b) there are studied the geodesics on generalized hyperbolic pants (a sphere with b boundary components and p cusps, with b+p =3) The results of the preceding paragraphs have allowed solving the problem of the behavior of geodesic in general case:

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