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# SET OF EQUILIBRIA IN MIXED-STRATEGY FOR HIERARCHICAL STRUCTURES

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**Abstract**: In most socio-economic entities a hierarchical structure can be distinguished. In the process of solving a task, the final result depends on the decisions made at each level. The choice made by a certain actor involved in solving the problem influences the choices of others and, not least, final profit. The paper aims to research mixed-strategy hierarchical games in three-level. That is, the game consists of three players, each of them has two strategies and a gain function. Players make moves in hierarchical mode: first player makes the choice and communicates the result to second player; second player knowing first player's choice, as well as third player's set of strategies and payoff function, makes his move and communicates the outcome to third player; finally, third player knowing the predecessors' choices, makes his choice. Thus, a situation is created and each player calculates his payoff. It is considered that all players maximize their payoff. The given model includes a wide range of problems that can appear in the socio-economic domain. To computing the Stackelberg equilibria set (SES), reverse induction and the graph reduction of best response mapping of the third player are used. A particular case of the results presented by Lozan and Ungureanu (2010, 2013, 2016, 2018) is studied and concretized. All possible cases for the graph of third player ($\mathbf{Gr_3}$) are investigated, the construction method is described by Ungureanu and Botnari (2005). Then, for player two, the possibilities that may arise for constructing his graph of best response mapping ($\mathbf{Gr_2}$) are analyzed. Finally, the first player determines his best moves on $\mathbf{Gr_2}$, thus determining the SES in mixed strategies.

**Keywords:** mixed-strategy, hierarchical game, graph of best response mapping, Stackelberg equilibrium.

**JEL Classification:** C02, C61, C62, C65, C72, C79.

## Introduction

The Stackelberg equilibria set (SES) may be identified by simplifying the graph of the best moves of the third player's via a set of optimization problems to the SES. The method of SES computing in mixed-strategy three-player hierarchical games is constructed. To build the graph of best response mapping the ideas of Ungureanu and Botnari (2005) are used. The paper investigates the notion of Stackelberg equilibrium by detailing/particularizing the theoretical theses presented by Ungureanu and Lozan (2008, 2010, 2013, 2016, 2018), in the case of games with three players. In works cited, you can see a more detailed list of references.

Consider a three-player hierarchical strategic game:

$$\boldsymbol{\Gamma} = \langle \mathbf{N}, \{\mathbf{S_p}\}_{p\in\mathbf{N}}, \{g_p(\mathbf{s})\}_{p\in\mathbf{N}} \rangle,$$

where:

- $\mathbf{N} = \{1, 2, 3\}$ is a set of players;
- $\mathbf{S_1} = \{1,2\}$, $\mathbf{S_2} = \{1,2\}$ and $\mathbf{S_3} = \{1,2\}$ are the sets of the strategies of the players;
- $g_p(\mathbf{s})$ is a utility function of player $p \in \mathbf{N}$ defined on the Cartesian product $\mathbf{S} = \mathbf{S_1} \times \mathbf{S_2} \times \mathbf{S_3}$;
- $\mathbf{s} = (s_1, s_2, s_3) \in \mathbf{S} = \mathbf{S_1} \times \mathbf{S_2} \times \mathbf{S_3}$, where $\mathbf{S}$ is the set of profiles.

Let's establish the matrix representation of the utility function $g_p(\mathbf{s})$, $p \in \mathbf{N}$

$$g_p(\mathbf{s}) = \mathbf{A_s} = \left[a^p_{s_1 s_2 s_3}\right]_{\mathbf{s} \in \mathbf{S}} \in \mathbf{R}^{\mathbf{2 \times 2 \times 2}}.$$

The mixed-strategy game corresponding to pure-strategy game is the following:

$$\mathbf{\Gamma}' = \langle \{1, 2, 3\}, \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}, \{f_1(\mathbf{x}, \mathbf{y}, \mathbf{z}), f_2(\mathbf{x}, \mathbf{y}, \mathbf{z}), f_3(\mathbf{x}, \mathbf{y}, \mathbf{z})\} \rangle,$$

where

- $\mathbf{X} = \{(x_1, x_2): x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$,
$\mathbf{Y} = \{(y_1, y_2): y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$ and
$\mathbf{Z} = \{(z_1, z_2): z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0\}$ are the sets of mixed strategies of the players;
- $f_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$, $f_2(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $f_3(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are the utility functions of the players defined on the Cartesian product $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ and

$f_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} a_{ijk} x_i y_j z_k,$

$f_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} b_{ijk} x_i y_j z_k,$

$f_3(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} c_{ijk} x_i y_j z_k.$

Assume that the players make their actions in a hierarchical manner:

– player 1 selects a strategy $\mathbf{x} \in \mathbf{X}$ and conveys it to player 2;
– player 2 chooses a strategy $\mathbf{y} \in \mathbf{Y}$ after observing the move $\mathbf{x}$ made by the player 1 and being aware of the set of strategies and the payoff function of the player 3. Subsequently, player 2 communicates both $\mathbf{x}$ and $\mathbf{y}$ to the player;
– player 3 selects a strategy $\mathbf{z} \in \mathbf{Z}$ after observing the moves $\mathbf{x}$ and $\mathbf{y}$ made by the preceding players.

Upon the formation of the profile $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, each player calculates the value of their respective cost function.

The player 1 has the leadership role for the players 2 and 3. The player 2 is positioned as the successor to the player 1 and the predecessor to the player 3. Player 3 is the successor for the players 1 and 2. When player $p$, $\boldsymbol{p} = \overline{\mathbf{1}, \mathbf{3}}$ makes a move, he possesses complete information about the leader choices, strategy sets and cost functions. However, he has no information about the choices made by the successor players. On the other hand, he has full information about the strategy sets and cost functions of the successor players. For simplicity, let's assume that all players maximize the values of their respective cost functions.

By reverse induction, player 3 establishes his optimal move mapping. Subsequently, the player 2 identifies his best move set on the third player's graph, and the player 1 computes his set of optimal moves on the set of player 2 (Ungureanu, 2008):

$$\mathbf{Br_3}(\mathbf{x},\mathbf{y}) = \mathrm{Arg}\max_{\mathbf{z}\in\mathbf{Z}} f_3(\mathbf{x},\mathbf{y},\mathbf{z}),$$

$$\mathbf{Br_2}(\mathbf{x}) = \mathrm{Arg}\max_{\mathbf{y},\mathbf{z}:\ (\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbf{Gr_3}} f_2(\mathbf{x},\mathbf{y},\mathbf{z}),$$

$$\hat{\mathbf{S}} = \mathrm{Arg}\max_{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbf{Gr_2}} f_1(\mathbf{x},\mathbf{y},\mathbf{z}),$$

where

$$\mathbf{Gr_3} = \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathbf{X}\times\mathbf{Y}\times\mathbf{Z}:\ \begin{matrix}\mathbf{x}\in\mathbf{X},\mathbf{y}\in\mathbf{Y},\\ \mathbf{z}\in\mathbf{Br_3}(\mathbf{x},\mathbf{y})\end{matrix}\right\},\ \mathbf{Gr_2} = \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathbf{Gr_3}:\ \begin{matrix}\mathbf{x}\in\mathbf{X},\\ (\mathbf{y},\mathbf{z})\in\mathbf{Br_2}(\mathbf{x})\end{matrix}\right\}.$$

Evidently, $\mathbf{Gr_2} \subseteq \mathbf{Gr_3}$.

**Definition 1.** Any profile $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \in \hat{\mathbf{S}}$ of the game is called Stackelberg equilibrium.

From the construction of the set $\hat{\mathbf{S}}$, the affirmation follows.

**Theorem 1.** The set $\hat{\mathbf{S}}$ of the Stackelberg equilibrium is non empty.

**Theorem 2.** If every strategy set $\mathbf{X}\subset\mathbf{R^2}$, $\mathbf{Y}\subset\mathbf{R^2}$, $\mathbf{Z}\subset\mathbf{R^2}$ is compact and every cost function $\boldsymbol{f_p}(\mathbf{x},\mathbf{y},\mathbf{z})$, $\boldsymbol{p} = \overline{\mathbf{1},\mathbf{3}}$ is continuous on its set of strategies and its successors, when the strategies of the predecessors are fixed and the corresponding best response set is compact, then the Stackelberg equilibria set $\hat{\mathbf{S}}$ is non empty.

*Proof.* The proof follows from the statement of the theorem and the Weierstrass theorem. □

## Main Results

Consider a 3-player mixed-strategy game $\mathbf{\Gamma}'$ formulated in section 1 whit the matrix:

$$A = (a_{ijk}), B = (b_{ijk}), C = (c_{ijk}), i = \overline{1,2}, j = \overline{1,2}, k = \overline{1,2}.$$

By substitutions:

$x_1 = x, x_2 = 1 - x, x \in [0;1];$

$y_1 = y, y_2 = 1 - y, y \in [0;1];$

$z_1 = z, z_2 = 1 - z, z \in [0; 1]$;

the equivalent normal form of the game $\mathbf{\Gamma}'$ was obtained:

$$\mathbf{\Gamma}'' = \langle\{1, 2, 3\}; \{[0; 1], [0; 1], [0; 1]\};\ \{u_1(x, y, z), u_2(x, y, z), u_3(x, y, z)\}\rangle,$$

where

$$u_1(x, y, z) = \Big(\big((a_{111} - a_{211} - a_{112} + a_{212} - a_{121} + a_{221} + a_{122} - a_{222})z + (a_{112} - a_{212} - a_{122} + a_{222})\big)y + (a_{121} - a_{221} - a_{122} + a_{222})z + (a_{122} - a_{222})\Big)x + \big((a_{211} - a_{212} - a_{221} + a_{222})z + (a_{212} - a_{222})\big)y + (a_{221} - a_{222})z + a_{222};$$

$$u_2(x, y, z) = \Big(\big((b_{111} - b_{121} - b_{112} + b_{122} - b_{211} + b_{221} + b_{212} - b_{222})z + (b_{112} - b_{122} - b_{212} + b_{222})\big)x + (b_{211} - b_{221} - b_{212} + b_{222})z + (b_{212} - b_{222})\Big)y + \big((b_{121} - b_{122} - b_{221} + b_{222})z + (b_{122} - b_{222})\big)x + (b_{221} - b_{222})z + b_{222};$$

$$u_3(x, y, z) = \Big(\big((c_{111} - c_{112} - c_{121} + c_{122} - c_{211} + c_{212} + c_{221} - c_{222})x + (c_{211} - c_{212} - c_{221} + c_{222})\big)y + (c_{121} - c_{122} - c_{221} + c_{222})x + (c_{221} - c_{222})\Big)z + \big((c_{112} - c_{122} - c_{212} + c_{222})y + (c_{122} - c_{222})\big)x + (c_{212} - c_{222})y + c_{222}.$$

Thus, $\mathbf{\Gamma}'$ is reduced to the game $\mathbf{\Gamma}''$ on the unit cube.

**Stage 1.** If the strategies of the first and second players is considered as parameters, then the third player has to solve a linear programming parametric problem:

$$u_3(x, y, z) \rightarrow max,\ z \in [0; 1] \qquad (1)$$

According to Ungureanu and Botnari (2005), the solution (1) is

$$\mathbf{Gr_3} = [0; 1]^3 \cap \{X_< \times Y_< \times 0 \cup X_= \times Y_= \times [0; 1] \cup X_> \times Y_> \times 1\},$$

where

$$X_< \times Y_< = \{(x, y): x \in [0; 1], y \in [0; 1], (\alpha_1 x + \alpha_3)y + \alpha_2 x + \alpha_4 < 0\},$$

$$X_= \times Y_= = \{(x, y): x \in [0; 1], y \in [0; 1], (\alpha_1 x + \alpha_3)y + \alpha_2 x + \alpha_4 = 0\},$$

$$X_> \times Y_> = \{(x, y): x \in [0; 1], y \in [0; 1], (\alpha_1 x + \alpha_3)y + \alpha_2 x + \alpha_4 > 0\},$$

$$\alpha_1 = c_{111} - c_{112} - c_{121} + c_{122} - c_{211} + c_{212} + c_{221} - c_{222},$$

$$\alpha_2 = c_{121} - c_{122} - c_{221} + c_{222},\ \alpha_3 = c_{211} - c_{212} - c_{221} + c_{222},\ \alpha_4 = c_{221} - c_{222}.$$

Depending on the values of $\alpha_1$, $\alpha_2$, $\alpha_4$ and $\alpha_4$, 59 cases are examined. As a result, 33 representations of the $\mathbf{Gr_3}$ graph are possible (see Figure 1).

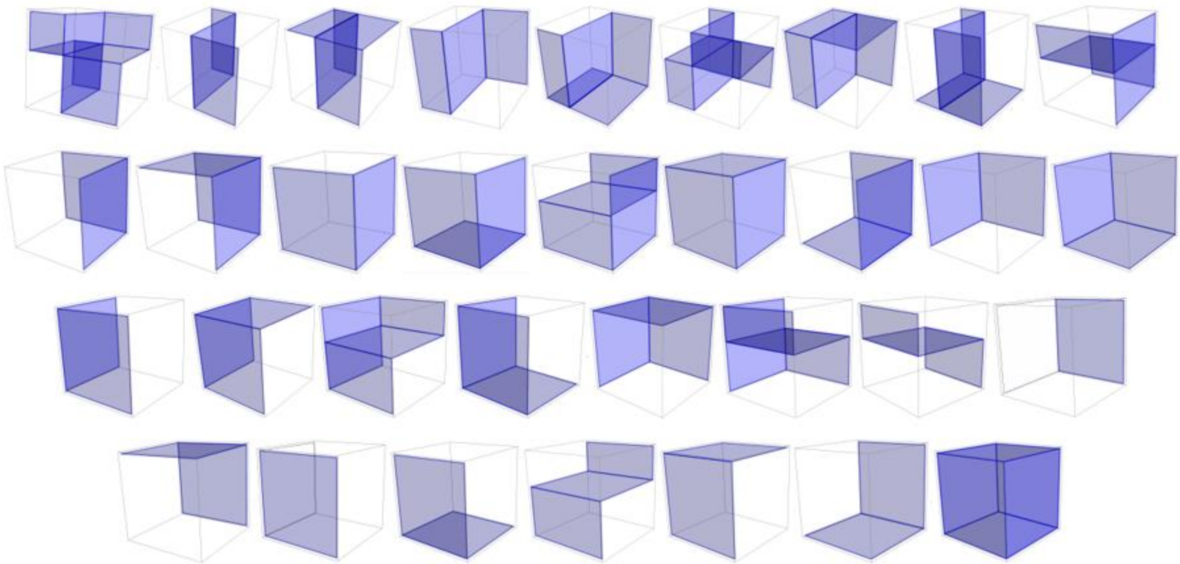

**Figure 1. The possible representations of $\mathbf{Gr_3}$**

**Source:** Realized by the author in Wolfram Mathematica

**Stage 2.** The second player solves a parametric optimization problem on $\mathbf{Gr_3}$:

$$u_2(x,y,z) \to max,\ (x,y,z) \in \mathbf{Gr_3} \qquad (2)$$

The problem (2) is equivalent whit three optimization problems of the form:

$$u_2(x,y,z) = (\beta_2 x + \beta_4)y + \beta_6 x + \beta_8 \to max,\ (y,z) \in [0;1]^3 \cap X_< \times Y_< \times 0,\ x \in [0;1];$$

$$u_2(x,y,z) = \big((\beta_1 z + \beta_2)x + \beta_3 z + \beta_4\big)y + (\beta_5 z + \beta_6)x + \beta_7 z + \beta_8 \to max,$$

$$(y,z) \in [0;1]^3 \cap X_= \times Y_= \times [0;1],\ x \in [0;1];$$

$$u_2(x,y,z) = (\beta_9 x + \beta_{10})y + \beta_{11} x + \beta_{12} \to max,\ (y,z) \in [0;1]^3 \cap X_> \times Y_> \times 1,\ x \in [0;1],$$

$\beta_1 = b_{111} - b_{121} - b_{112} + b_{122} - b_{211} + b_{221} + b_{212} - b_{222}$, $\beta_2 = b_{112} - b_{122} - b_{212} + b_{222}$,

$\beta_3 = b_{211} - b_{221} - b_{212} + b_{222}$, $\beta_4 = b_{212} - b_{222}$, $\beta_5 = b_{121} - b_{122} - b_{221} + b_{222}$,

$\beta_6 = b_{122} - b_{222}$, $\beta_7 = b_{221} - b_{222}$, $\beta_8 = b_{222}$, $\beta_9 = b_{111} - b_{121} - b_{211} + b_{221}$,

$\beta_{10} = b_{211} - b_{221}$, $\beta_{11} = b_{121} - b_{221}$, $\beta_{12} = b_{221}$.

When the second player maximizes his gain function on each non-empty component of $\mathbf{Gr_3}$, the components can be divided into at most 5 parts. The values of the utility function $u_2(x,y,z)$ is evaluated across all component parts, and the highest value is recorded. The set $\mathbf{Gr_2}$ comprises all the components where the optimal values of the function $u_2(x,y,z)$ are achieved, along with the best strategies of the players 2 and 3: $y$ and $z$, corresponding to the saved parts. In the process of building $\mathbf{Gr_2}$ on each component of $\mathbf{Gr_3}$, 9/13/15 cases are possible depending on the $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$, $\beta_7$, $\beta_8$, $\beta_9$, $\beta_{10}$, $\beta_{11}$ and $\beta_{12}$ values. On $z$=*0* component, when $\mathbf{Gr_3}$ is the first representation in Figure 1, one of the cases illustrated in Figure 2 is possible.

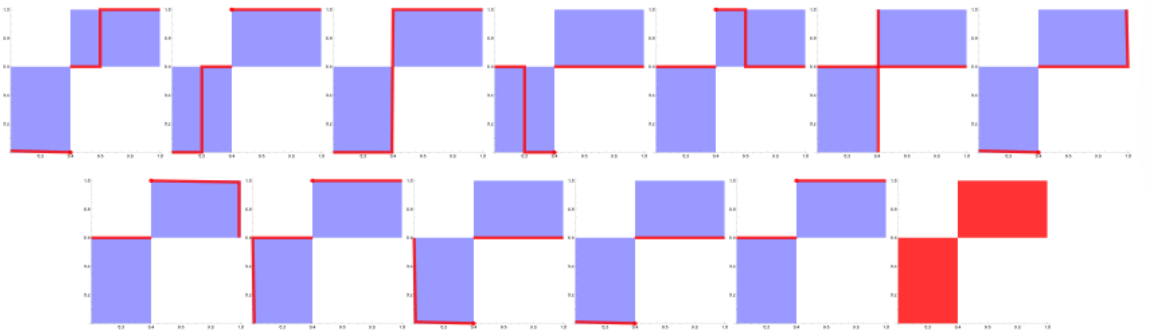

**Figure 2. The possible representations on the $z = 0$ component, case 1 in Figure 1.**

**Source:** Realized by the author in Wolfram Mathematica.

**Stage 3.** The first player calculate his best moves on each components of the $\mathbf{Gr_2}$:

$$\mathbf{SES}(\mathbf{\Gamma}'') = \mathrm{Arg}\max_{(x,y,z)\in\mathbf{Gr_2}} u_1(x,y,z).$$

The following notations are considered:

$$\hat{S}_k = \max_{(x,y,z)\in\mathbf{Gr_2}(k)} u_1(x,y,z),\ \mathbf{Gr_2}(k) \text{ is the } k \text{ part of } \mathbf{Gr_2};$$

$$\hat{\mathbf{S}} = \max_k \hat{S}_k.$$

He determines the corresponding values simultaneously comparing them with preceding value and the best is saved in the result, the Stackelberg equilibria set is established.

## 1. Example

In this paragraph an example will be solved to better understand the method of determining the set of Stackelberg equilibria.

Matrices of the three person game are:

$$a_{1**} = \begin{bmatrix} 2 & 5 \\ 5 & 1 \end{bmatrix},\ a_{2**} = \begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix};$$

$$b_{*1*} = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}, b_{*2*} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix};$$

$$c_{**1} = \begin{bmatrix} 12 & 0 \\ 0 & 2 \end{bmatrix}, c_{**2} = \begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix}.$$

Let's identify the set of Stackelberg equilibria. The mixed strategy normal form game on the unit cube has the cost functions:

$u_1(x, y, z) = (y + z - 1)x - 7yz + 3y + 2z + 2;$

$u_2(x, y, z) = (7xz - 6x - 5z + 4)y - 2xz + 3x + z + 1;$

$u_3(x, y, z) = (24xy - 6y - 8x + 2)z - 10xy + 6x + 4y.$

The sketched method applies.

**Stage 1**. $\mathbf{Gr_3}$ is determined.

We have the case $\alpha_1 > 0$, $\alpha_3 < 0$, $\alpha_1 > -\alpha_3$, $\alpha_4 > 0$, $\alpha_4 < -\alpha_3$, result:

$$X_< \times Y_< \times 0 = \begin{cases} x \in \left[0, \frac{1}{4}\right), y \in \left[0, \frac{1}{3}\right], z = 0, \\ x = \frac{1}{4}, y \in [0, 1], z = 0, \\ x \in \left(\frac{1}{4}, 1\right], y \in \left[\frac{1}{3}, 1\right], z = 0. \end{cases}$$

$$X_= \times Y_= \times [0, 1] = \begin{cases} x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z \in [0, 1], \\ x = \frac{1}{4}, y \in [0, 1], z \in [0, 1], \\ x \in \left(\frac{1}{4}, 1\right], y = \frac{1}{3}, z \in [0, 1]. \end{cases}$$

$$X_> \times Y_> \times 1 = \begin{cases} x \in \left[0, \frac{1}{4}\right), y \in \left[\frac{1}{3}, 1\right], z = 1, \\ x = \frac{1}{4}, y \in [0, 1], z = 1, \\ x \in \left(\frac{1}{4}, 1\right], y \in \left[0, \frac{1}{3}\right], z = 1. \end{cases}$$

So, $\mathbf{Gr_3} = [0; 1]^3 \cap \{X_< \times Y_< \times 0 \cup X_= \times Y_= \times [0; 1] \cup X_> \times Y_> \times 1\}$. Figure 3 shows with blue the graph of best response mapping of third player.

**Stage 2.** $\mathrm{Gr}_2$ is determined, maximize $u_2(x, y, z)$ on $\mathrm{Gr}_3$.

We have the cases $\beta_2 < 0$, $\beta_4 > 0$, $-\beta_2 > \beta_4$ and $\beta_9 > 0$, $\beta_{10} < 0$, $\beta_9 = -\beta_{10}$. In $y = \frac{1}{3}$ is the case $x \in [0;1]$ and $z = 0$.

On the component $X_< \times Y_< \times 0$ is obtained:

$$\begin{cases} x + \frac{7}{3}, & x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z = 0, \\ -3x + 5, & x \in \left[\frac{1}{4}, \frac{2}{3}\right), y = 1, z = 0, \\ 3, & x = \frac{2}{3}, y \in \left[\frac{1}{3}, 1\right], z = 0, \\ x + \frac{7}{3}, & x \in \left(\frac{2}{3}, 1\right], y = \frac{1}{3}, z = 0. \end{cases}$$

On the component $X_< \times Y_< \times [0,1]$ is obtained:

$$\begin{cases} x + \frac{7}{3}, & x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z = 0, \\ \frac{17}{4}, & x = \frac{1}{4}, y = 1, z = 0, \\ x + \frac{7}{3}, & x \in \left(\frac{1}{4}, 1\right], y = \frac{1}{3}, z = 0. \end{cases}$$

On the component $X_< \times Y_< \times 1$ is obtained:

$$\begin{cases} \frac{4}{3}x + \frac{5}{3}, & x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z = 1, \\ x + 2, & x \in \left[\frac{1}{4}, 1\right), y = 0, z = 1, \\ 3, & x = 1, y \in \left[0, \frac{1}{3}\right], z = 1. \end{cases}$$

After comparing the $u_2(x, y, z)$ values on all parts of the components and saving the best one, $\mathbf{Gr_2}$ may be represented as:

$$\mathbf{Gr_2} = \begin{cases} x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z = 0, \\ x \in \left[\frac{1}{4}, \frac{2}{3}\right), y = 1, z = 0, \\ x = \frac{2}{3}, y \in \left[\frac{1}{3}, 1\right], z = 0, \\ x \in \left(\frac{2}{3}, 1\right], y = \frac{1}{3}, z = 0. \end{cases}$$

Figure 3 shows with green the graph of best response mapping of second player.

**Stage 3.** The **SES** is determined. So, the components of $\mathbf{Gr_2}$ may be used to determine the equilibria sets.

For, $\mathbf{Gr_2}(1)$: $x \in \left[0, \frac{1}{4}\right), y = \frac{1}{3}, z = 0,$

$\widehat{\boldsymbol{S}} = \hat{S}_1 = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(1)} u_1(x,y,z) = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(1)} \left(\frac{2}{3}x + 3\right) = \frac{7}{6}$, in $\left(\frac{1}{4}, \frac{1}{3}, 0\right)$, but $\frac{1}{4} \notin \left[0, \frac{1}{4}\right)$, result $\mathbf{SES} = \emptyset$.

For, $\mathbf{Gr_2}(2)$: $x \in \left[\frac{1}{4}, \frac{2}{3}\right), y = 1, z = 0,$

$\widehat{\boldsymbol{S}} = \hat{S}_2 = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(2)} u_1(x,y,z) = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(2)} (5) = 5$, $\mathbf{SES} = \left\{\left(\frac{1}{4} \le x < \frac{2}{3}, 1, 0\right)\right\}$.

For, $\mathbf{Gr_2}(3)$: $x = \frac{2}{3}, y \in \left[\frac{1}{3}, 1\right], z = 0,$

$\hat{S}_3 = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(3)} u_1(x,y,z) = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(3)} \left(\frac{11}{3}y + \frac{4}{3}\right) = 5$ in $\left(\frac{2}{3}, 1, 0\right)$, result $\hat{S}_3 = \widehat{\boldsymbol{S}}$ and $\mathbf{SES} = \left\{\left(\frac{1}{4} \le x \le \frac{2}{3}, 1, 0\right)\right\}$.

For, $\mathbf{Gr_2}(4)$: $x \in \left(\frac{2}{3}, 1\right], y = \frac{1}{3}, z = 0,$

$\hat{S}_4 = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(4)} u_1(x,y,z) = \max\limits_{(x,y,z)\in\mathbf{Gr_2}(4)} \left(\frac{2}{3}x + 3\right) = \frac{11}{3}$ , $\widehat{\boldsymbol{S}} > \frac{11}{3}$ and the **SES** remains unchanged.

Solution of the game is $\mathbf{SES} = \left\{\begin{pmatrix} \frac{1}{4} \le x \le \frac{2}{3} \\ 1 - x \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ whit the gain $\left(5, \left[3, \frac{17}{4}\right], \left[\frac{4}{3}, 3\right]\right)$.

In figure 3, the set of Stackelberg equilibria in mixed strategies of the game is represented in red.

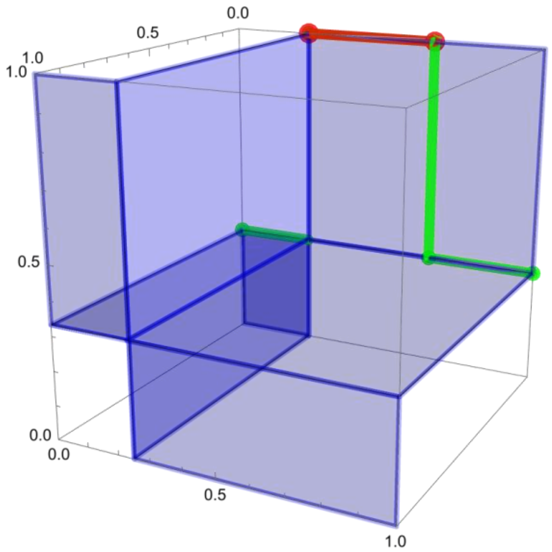


**Figure 3. The $\mathbf{Gr_3}$, $\mathbf{Gr_2}$ and SES**

**Source:** Realized by the author in Wolfram Mathematica

## Conclusions

The hierarchical mixed-strategy game of three-player is analyzed. In the most difficult case, the **SES** may be partitioned into a maximum of six parts; on each of them the Stackelberg equilibrium are calculated by comparing the optimal values of the cost function of first player on the convex parts of the graph of optimal moves mapping of the player 2. The **SES** consists of the component points on which the best values were achieved. To build $\mathbf{Gr_3}$, 59 cases are investigated as a result 33 representations are obtained. The possible results for determining $\mathbf{Gr_2}$ on each component of $\mathbf{Gr_3}$ are analyzed. Of course, to simplify the pure and mixed strategy game and improve the method, the equivalent dominant and dominated strategies in the pure game can be identified.

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