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TYPICAL GEODESICS ON HYPERBOLIC MANIFOLDS OF DIMENSION 2

Vladimir BALCAN, PhD
Email: vl_balkan@yahoo.com
Academy of Economic Studies of Moldova,
Republic of Moldova, Chisinau, 61, Banulescu Bodoni Street
Phone: (+373 22) 22 41 28, www.ase.md

Abstract: *Let M be a complete hyperbolic surface of genus g, with k punctures and n boundary geodesics. In this paper we investigate typical behavior of geodesics for some hyperbolic 2-manifolds, and discuss some extension of those results to the case of a arbitrary hyperbolic surfaces(on a closed orientable hyperbolic surface M of genus g at least 2, in the case of non-compact hyperbolic surface and for a compact hyperbolic surface with non-empty boundary).*

Keywords*: behavior of geodesics, the multilateral, the method of colour multilaterals, hyperbolic right angled hexagon, hyperbolic right angled octagon pair pants (meaning surfaces of signature (0,3)). hyperbolic surface with genus g, k puncture and n geodesic boundaries.*

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This paper focuses on the problem of classification (the behavior) of geodesics on hyperbolic 2 dimensional manifolds. Geodesics are those curves on the surface that are not geodesically curved. Considering their role on surfaces they can be compared to straight lines in plane and they are called "the most straight lines" on surface. One of the main approaches to study of the geometry of manifolds and understand its structure is through the investigation of geodesics, the shortest path or geodesic between the two points. Behavior of geodesics in hyperbolic surfaces has been a fruitful subject of research for many years. Such geodesics are often studied by looking at their lifts in covering spaces of the surface. The chaotic behavior of geodesics on surfaces of constant negative curvature and finite volume has been known since Hadamard (1898). In this work for the first time systematically is described and summarize the results of our study the geometry of behavior of the geodesics on hyperbolic manifolds of dimension 2. These results would be considered as an analogue of the coding of geodesics on the modular surface in terms of continued fraction expansions. The study of global behavior of the geodesics is one of the main topics in geometry and their behavior under variations helps us to understand the geometry of hyperbolic manifolds.

The definitions of geodesic lines in various spaces depend on the particular structure (metric, line element, linear connection) on which the geometry of the particular space is based. In the geometry of spaces in which the metric is considered to be specified in advance, geodesic lines are defined as locally shortest. The local behavior of geodesic curves is similar to that of straight lines in Euclidean space. A sufficiently short arc of a geodesic line is the shortest among all rectifiable curves with the same ends. Only one geodesic line passes through any point in a given direction. Families of geodesic lines, considered as possible trajectories of motion, form a subject of the theory of dynamical systems and ergodic theory.

Thanks to the development of the new constructive approach, in this paper, the author succeeded to receive "in a certain sense" the solution for the behavior of the geodesics in general on the hyperbolic manifolds of dimension 2, structure of geodesics and their types. In order to discuss the results of this work, it will be necessary to agree on some definitions of the basic concepts. The following terminology will be used regularly throughout this paper. Let M be a finite area hyperbolic surface. Then M is homeomorphic to a closed surface with finitely many points removed. Each of these points called punctures, has special neighborhoods in M called cusp. A geodesic in a hyperbolic manifold is a locally distance - minimising curve, and is said to be simple if it has no transverse self-intersections (there-fore it is either an embedded copy of R or an embedded circle) and non simple otherwise. A geodesic on surface M is said to be complete if it is not strictly contained in any other geodesic, i.e., it is either closed and smooth, or open and of infinite length in both directions. Complete geodesics coincide with those which never intersect ∂M . Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp. Throughout, we use the term geodesic to refer to a complete infinite geodesic; a geodesic ray is a half-infinite ray; finally, a geodesic arc is a finite segment lying along some geodesic (which we assume to be closed unless otherwise stated). For a hyperbolic surface M some of the geodesics γ will come back to the point they start and fit in a smooth way. These are called closed geodesics. It ends up that there are finitely many closed geodesics of a given lenght (if any). Geodesics on smooth surfaces are the straightest and locally shortest curves. A hyperbolic surface is a surface which constant negative curvature. Unlike the plane, which is flat, or the sphere which has positive curvature, these surfaces are negatively curved. On a hyperbolic surface, some geodesics are infinitely long, like straight lines in the plane, but others close up into a loop, like the great circle on a sphere. Two basic properties are responsible for their importance: first, that from any point of a manifold there starts a unique geodesic in any direction. Second, the length minimization property (connecting two given points on a manifold with a locally shortest curves). On smooth surfaces geodesics possess both properties. Geodesics on hyperbolic surfaces are briefly discussed in [Balcan V., 2017, p.191].

The main objective of the article is to describe the qualitative behavior of the geodesics on 2-dimensional hyperbolic manifolds. A (closed) hyperbolic surface can be defined either by a Riemannian metric of constant negative curvature or (thanks to the uniformization theorem) by a quotient of hyperbolic plane by a discrete group of isometries, isomorphic to the fundamental group of the initial surface, acting properly discontinuously on hyperbolic plane. A hyperbolic surface of genus g with k punctures and n holes and with no boundary is said to be of type (g, k, n). Such surfaces are said to be of finite type. A standard tool in the study of compact Reimann surfaces is the decomposition into "pairs of pants" (Y pieces). Given a surface of genus $g \geq 2$, there are $3g-3$ simple closed pairwise non-intersecting geodesics which partition the surface into $g-1$ such pieces. A Riemann surface of signature (g, n) is an oriented, connected surface of genus g with n boundary components, called boundary geodesics, which is equipped with a metric of constant negative curvature. Here by a geodesic we always mean a locally shortest curve.

We want to describe their global behavior: a) when are geodesics closed ? b) when are the dense ? c) quantitatively, how do they wrap around the surface ? These questions admit notably precise answers, as we are going to see. Much less is known about the behavior of geodesics on hyperbolic surfaces. How do geodesics on the hyperbolic surface behave or how can we determine the behavior of a given geodesic on the hyperbolic surface? The qualitative behavior of geodesics on even seemingly simple hyperbolic surfaces can be surprisingly complex. A nother method, of arithmetic nature, uses continued fraction expansions of the end points of the geodesic at infinity and is even older – it comes from the Gauss reduction theory. Introduced to dynamics by E. Artin in a 1924 paper, this method was used to exhibit dense geodesics on the modular surface. The problem of understanding the geometry and dynamics of geodesics and rays (i.e. distance-minimizing half geodesics) on hyperbolic manifolds dates back at least to Artin, who started to study the qualitative behavior of geodesics on hyperbolic surfaces. Artin studied these questions by cleverly encoding geodesics using continued fractions.

We investigate in detail the global behavior of the geodesics on the simplest hyperbolic surfaces: hyperbolic horn (funnel end), hyperbolic cylinder and parabolic horn (cusp, horn end), or parabolic cylinder. The problem of behavior of geodesic is solvable for a hyperbolic surface called hyperbolic horn (funnel). A hyperbolic horn (funnel end) is a two-dimensional manifold, obtained from the strip between the two parallel straight lines of the hyperbolic (Lobachevsky) plane by matching the border lines by shifting (sliding), its axis being parallel to he border lines and beyond the strip between them. The funnel, i.e. the factor-space $H^2_+\backslash\Gamma$, is an (open) half of the hyperbolic cylinder. The border circumference does not belong to that half and there for the surface of the hyperbolic horn is incomplete. The funnel is half of the hyperbolic cylinder, bounded by their closed geodesic. The full funnel continues to flare out exponentially and has infinite area. There is a

Theorem 1. On the funnel the problem of behavior of a geodesic is solvable.

The theorem is resolved using the affirmations I-IV set out below.

It is clear that the hyperbolic horn (funnel), i.e. the factor space H^2/Γ, is the open «half» of the cylinder C considered above where the border circumference a' does not belong to that half, therefore the funnel is incomplete surface.

Affirmation I. There are no closed geodesics on the funnel.

Affirmation II. If the geodesic l on the funnel $M^2=H^2_+/\Gamma$ is defined so that its covering lies on a straight line intersecting the line a, then the *geodesic l is infinite without self-intersections* and any of its points divides it into two rays: one ray of finite length, another ray of infinite length.

Affirmation III. If the covering l' for the geodesic l for the funnel M^2 is the straight line parallel to the line a, then the *geodesic l is infinite, without self-intersections points*, and any of its points divides it into two congruent rays.

Affirmation IV. If the covering l' for the geodesic l is a straight line divergent with the axis of shifts, then the geodesic l is *infinite* and it has only a finite number k of double self-intersection points.

Here in none of the cases the geodesic was not a closed one, as said in the Affirmation I. Therefore, in each of the three possible cases the behavior of geodesic is fully described, and since any other cases are impossible, it has been demonstrated that the behavior of geodesic on hyperbolic funnel is fully solvable. So, every geodesic curve γ on the hyperbolic horn is of one of the four types or the following types of geodesic on funnel are identified: 1) *there are no closed geodesics*; 2) there is a geodesic of *infinite length, without self-intersections points*, and any of its points divides the geodesic into two rays: one ray of finite length and another ray of infinite length; 3) there is an *infinite geodesic, without self-intersections points* and any of its points divides it into two congruent rays; 4) *there is an infinite geodesic and it has a finite number k* of double self-intersection points *and they are all divisible by* 2. The number k of self-intersection points of an examined geodesic is equal to p.

The problem of behavior of a geodesic on a hyperbolic cylinder is solvable. One may define the *hyperbolic cylinder* as a non-compact two-dimensional manifold obtained from the strip from between the two divergent lines of the hyperbolic (Lobachevsky) plane by identifying the divergent border lines by shift (sliding), its axis being a common perpendicular for the said border lines, its shift being equal to the length of such translation. The factor space $H^2\backslash\Gamma$ is a some kind of cylindrical surface also called hyperbolic cylinder. The hyperbolic cylinder is the union of two funnels.

Theorem 2. On the hyperbolic cylinder C = Ц^2=H^2/Γ the geodesic's behavior problem is solvable.

The proof of theorem comes from the affirmations I and II set out below.

Affirmation I. There are no closed geodesics on the cylinder C (both simple, different from the narrow geodesic core of cylinder and non-simple ones).

This results from the fact that the closed geodesics $\tilde{b}$ correspond to the translation $\vec{b}$. But such translation should transform into itself the straight line *a*, while this is possible only when the line *b* is on the line *a*, i.e. it is a translation along the line *a.* This translation along the line *a* on a hyperbolic cylinder will lie on a geodesic core (the narrowest place of cylinder). It is the only simple close geodesic on that surface.

*Affirmation II***.** If the geodesic's image intersects the straight line *a*, such a geodesic is *a geodesic without self-intersection points, infinite in both directions* (at both ends).

Let us consider the behavior of geodesic on a parabolic cusp (parabolic cylinder). We shall call a *parabolic horn (cusp)* the two-dimensional manifold obtained from the strip from between the two parallel lines of the hyperbolic (Lobachevsky) plane by identifying the border lines by horocyclic rotation determined by these lines. The parabolic cylinder is a special case (its small end is a cusp, while the "horn" end carriers the hyperbolic metric). There appears the

Theorem 3. The problem of behavior of a geodesic on a horn end (cusp) is solvable.

The study of universal cover of parabolic cusp demonstrates that:

1) If the arbitrary straight line *c* does not cross the obstructing line of the pair determining the horocyclic rotation *w* and identified upon that rotation, the image of the said straight line on this surface(cusp) is isometric to the usual straight line of a hyperbolic surface *(simple infinite length, without self-intersection)*;

2) If the image of the geodesic *c* on the hyperbolic plane H^2 is a straight line intersecting the said geodesic and if it is different from the obstructing straight line, then the *geodesic c* is *infinite in both directions* (at both ends) and it has only a finite number k of double self-intersection points. In the particular case, both ends of the geodesic can go to the some point at infinity.

3) *There are no closed geodesics on the parabolic cusp,* because no translation in the group $\Gamma = \langle w \rangle$.

The study of the geodesics on hyperbolic surfaces can be reduced to the study of the curves on a hyperbolic pair of pants. Compact hyperbolic surfaces can be seen as an elementary pasting of geodesic polygons of the hyperbolic plane. Conversely, cutting such a surface along disjoint simple closed geodesics (a partition), one obtains a family of pair of pants (surfaces of signature (0,3)), which in turn can be readily cut to obtain a pair of isometric right-angled hexagons. Let M be a surface and let P be a pair of pants. We focus on getting the behavior of geodesics on a hyperbolic pair of pants *P*. As a direct consequence we get the behavior of geodesics on any surface *M*. We do this as follows. First, there is a unique way to write *P* as the union of two congruent right-angled hexagons. Take this decomposition (see on Fig.1).

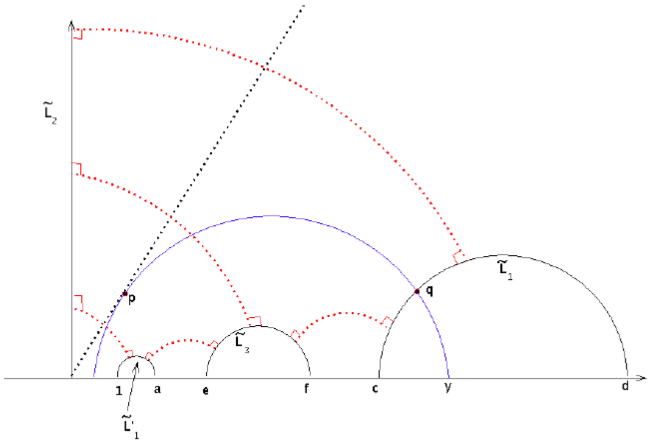

Fig. 1 Universal cover of P in H^2

We examining different types of behaviors exhibited by geodesics on a given pair of hyperbolic pants (general, symmetric and generalized) and study infinite simple geodesic rays and complete geodesics Three simple, pairwise non-intersecting bi-infinite geodesics in the hyperbolic pair of pants, that each “spiral” towards two different boundary components of pair of pants (see Fig, 2).

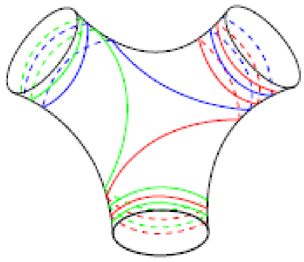

Fig. 2 Curves on pair of pants

We also allow the degenerate case in which one or more of the lenths vanish. We call a generalized pair of pants a hyperbolic surface which is a homeomorphic to a sphere with three holes, a hole being either a geodesic boundary component or a cusp. Symmetric hyperbolic pairs of pants, that is, hyperbolic pair of pants which have three geodesic boundary components of equal lenths.

For the behaviour of the geodesics on the specified fragments (hyperbolic pants, etc.) it is used a certain figure, named in the text of the work *the multilateral.* The multilateral is obtained from a right angle hexagon as follows. We construct a hyperbolic hexagon with right angles on the hyperbolic plane H^2. For a certain value $r = r_1$ of the radius of the circle, inscribed in the right angle triangle, this triangle becomes limited: its vertices become infinitely remote points, and the sides - in pairs parallel lines. It is to be noted that these triangles decompose H^2. If we continue to increase the radius of the inscribed circle, then the sides of the triangle become pairwise divergent straight lines, the vertices - are ideal points and the area of the triangle is infinite. From the hyperbolic geometry we have that for given two disjoint geodesics on the plane H^2 with four different end points at the infinity (divergent), there is only one geodesic perpendicular to both. But, if from the obtained “ beyond the limit” triangle we cut off the "excess" endless pieces with the help of common perpendiculars of the pairs of its sides, then we get an *equiangular-semi-regular hexagon* (see Fig. 3). All the angles of this hexagon are straight and the sides over one are equal. Symmetric right-angled hyperbolic hexagons, that is, convex right –angled hyperbolic hexagons having three non-adjacent edges of equal length.

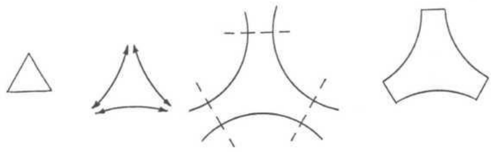

Fig. 3 Orthogonal section of a triangle and obtaining a hexagon with right angles

We'll call the newly appeared sides black, and „remnants" of the sides of the original triangle are white: than we can say that all the white sides are equal to each other in pairs, and all the black sides are also equal pairwise to each other, and the angle between the intersecting white and black sides is straight. Thus the resulting hexagon is a Coxeter, and the group generated by reflections in its sides is a Coxeter group. But before building all this Coxeterian partition, it is very useful to first make reflections only on the black sides of the hexagon and on their images, obtained by such reflections. Continuing indefinitely the reflection in these black sides, we get some new kind of regular polygon (it would be more accurate to say - *the multilateral*). The sides of this „ *multilateral*" (without vertices and angles) are straight lines, tangents (regular) system of circumferences on the hyperbolic plane H^2 (see Fig.4). Obviously, the reflections in the sides of the straight lines of this multilateral can cover the whole (entire) hyperbolic plane H^2 (Fig. 4). To facilitate the understanding and further description, we agree to call the sides of the six-rectangle (right angled hexagon) black, if they are obtained from boundary geodesic circles of pants, and the other three sides we agree to consider painted in different colours (for example, red, blue and green straight). Exactly, this figure is also called in the work as a multilateral (in contrast to the polygon, the figure has no vertices and angles, hence its name - the multilateral). The study of the behavior of the geodesics in this paper is being carried out gradually, in order of collecting the surface, the reverse order of cutting the surface into fragments (i.e. pants). The surface is cut into typical pieces (for example, on pants or their degenerations, on right hexagons, etc.) and the question of the behavior of the geodesics for each piece is solved on it, and then the result of the investigation returns (by gluing) onto the original surface.

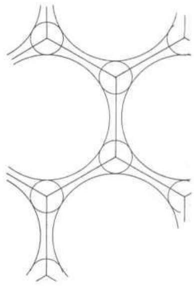

Fig. 4 A generalized regular multilateral, described near the regular system of congruent circumferences.

With the help of these multilaterals, it is possible to determine the nature of the behavior of the geodesics on the surface, not more complicated than how Artin studied the global behavior of geodesics on hyperbolic surfaces by cleverly encoding geodesics using continued fractions. Any given hyperbolic (closed, i.e., ordinary) surface can be cut into pants and the question is how, when gluing such pants, connect them on a common surface. But it may seem (when gluing of the surface from the pants is not finished yet) that the surface of genus g has also n components (the surface has a geodesic boundary). And, going further, we notice that the boundary of the surface can degenerate: transform into cuspidal ends (cusps) and into conical points. Thus, we arrive at the most general case, the surfaces of the signature (g, n, k), the preliminary investigation of the behaviour of the geodesics on these pieces. To summarize what has been said, we can conclude that a concrete method of investigating the behavior of the geodesics on hyperbolic 2-manifolds is based on the idea of preliminary research on these pieces (on the set of hyperbolic pants and their degenerations), in the subsequent consolidation of research results using the method proposed in this paper (sometimes called the method of generalized coloured multilaterals). The main purpose of this article is to indicate an algorithm (the construction of a practical approach) that allows determine the behavior of this geodesic on hyperbolic manifolds. Also the aim is to obtain new results in following areas: a) the solution of the question of the qualitative behavior of the geodesics in general (if a point and the direction of the tangent at that point are given) on 2-dimensional hyperbolic manifolds; b) a new method for solving the problem of the behavior of the geodesics on hyperbolic manifolds is developed - *the method of colour multilaterals*; c) with the help of this technique, the question of the qualitative behavior of the geodesics in general on hyperbolic 2-manifolds is solved. In more detail, the following main results of the study were obtained. A new constructive method for investigating the global behaviour of the geodesics on hyperbolic manifolds (the method of colour multilaterals) is given in this paper. The solution is based on the study of the behavior of the geodesics on the simplest hyperbolic surfaces (hyperbolic pants, degenerate hyperbolic pants, thrice-punctured sphere, etc.), some of which have long attracted the attention of geometers. In this paper is used the Këbe method of geodesic cutting of hyperbolic 2-manifolds into hyperbolic pants with a nonempty boundary (edge). In hyperbolic geometry, hyperbolic right angled hexagons are used as a tool for analysing the behaviour of the geodesic (and surfaces). The discrete group Γ is defined in the usual way via its fundamental domain F (glued from the proper number of right angled hexagons). Hyperbolic pants are the only compact hyperbolic surfaces with a geodesic boundary that can't be simplified by cutting along closed simple geodesics. In fact, any pants with boundary geodesics are uniquely determined by the length of their boundary geodesics, because any hyperbolic right angled hexagon is uniquely defined by three alternating (non-adjacent) lengths of sides that can be arbitrarily set. We consider the universal covering of hyperbolic pants (the hyperbolic plane H^2) and lines that cover a given geodesic. Let it be on H^2 a right angled hexagon H and let H' denote its image under reflection from the side of δ_{13} (Fig. 3). When identifying the corresponding sides δ_{12} and δ_{12}, as well as δ_{23} and δ_{23} of this right angled geodesic octagon (Fig. 5), we obtain hyperbolic pants *P* with boundaries $\alpha_1, \alpha_2, \alpha_3$. As the fundamental region for the corresponding pants *P* we choose this hyperbolic right angled octagon (the double of a right angled hexagon H in the plane H^2).

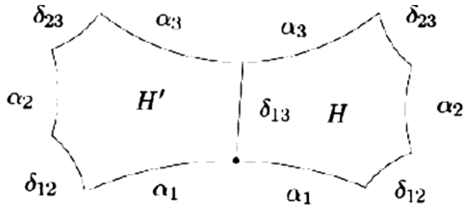

Fig. 5 The hexagon H and its reflection at the side δ_{13} is a symmetrical hexagon H'

This geodesic right angled octagon is the fundamental domain of the group $\Gamma = \langle t_1, t_2, t_3 \rangle$ generated by hyperbolic translations t_1, t_2, t_3, (the pants can be obtained by factorizing the hyperbolic plane H^2 by a discrete co-compact group Γ generated by translations t_1, t_2, t_3, where the translation t_i is determined by the vector $2 \cdot \alpha_i, i = 1, 2, 3$). To describe the behaviour of an arbitrarily given (some) geodesic on hyperbolic pants emanating from the point A in a given direction, we need to consider how the direct, covering this geodesic, behaves on the universal cover of these pants. In other words, how this straight line is located relative to the sides of α_3 , α_1 , α_2 of the hyperbolic right angled octagon (the so-called "colour" straight - blue, green, red). Walking along hyperbolic octagon, we can't cross the boundary components α_3 , α_1 , α_2 (" coloured circumferences "), but we can pass through the sides δ_{13} , δ_{12} , δ_{23} of a hyperbolic hexagon (the so-called "black" sides). Along with the coloured sides, the categories of coloured angles are built. A pair of "adjacent" colour angles uniquely determines the next colour angle with the help of colour (coloured "straight lines - blue, green, red) or with the help of geodesic sides α_1 , α_2 , α_3. Suppose, for the beginning, point A is fixed on the surface of hyperbolic pants, and we need to understand how the geodesic's behaviour depends on the direction (from the directing vector emanating from point A). In this situation each side determines the angle of its colour with the vertex in the point A (and the sides parallel to the colour side) and in each category of angles it is uniquely determined which sides δ_{13} , δ_{12} , δ_{23} (or "black" sides) it is necessary to cross to be within the scope of the corresponding colour side. Thus, on hyperbolic pants, the problem of the behaviour of any geodesic passing through a fixed point is uniquely solvable by the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral obtained from a right angled hexagon. Thus, *on hyperbolic pants is the problem of the behaviour of any geodesic* that passes through a fixed point and is uniquely solvable with the help of *the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral*, obtained from a right angled hexagon. Further, the concept of the category of angles is introduced, and with the help of these categories an algorithm for recognizing the type of a geodesic is given.

Main results of the present work are as follows. In the work is given a new constructive method (a new approach) for solving the problem of the behavior of geodesic on a arbitrary hyperbolic surfaces of signature (g, n, k), i.e., method allowing to answer the question about the structure on the global of examined geodesic at its indefinitely extension (geodesics can be extended indefinitely) on both directions. Such a compressed formulated result can be disclosed as follows. For this purpose, with the help of proposed practical approach at first are studied geodesics at the simplest hyperbolic manifolds: 1) it is solved the problem of the behavior of geodesic on the simplest hyperbolic surfaces (hyperbolic horn; hyperbolic cylinder; parabolic horn (cusp)); 2) it is investigated and described the behavior of the geodesic lines on hyperbolic surfaces of signature (0,3) (hyperbolic pants); it is found special case: behavior of ortho-boundary geodesics and orthogeodesics, and their general structure, i.e., it is obtained classification of geodesics launched (emanating) normally from the point of geodesic boundary of pants (see Fig. 5). Is said to be orthogeodesic - a geodesic segment perpendicular to the boundary at its initial and terminal points.

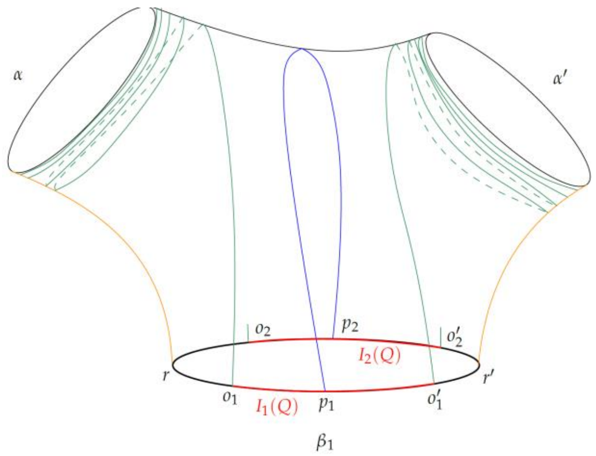

Fig. 5 Pants *P* with spiraling geodesics

Investigation of behavior of the geodesics on the listed above surfaces, allowed finding answer of assigned task in general case: 3) it is investigated and found behavior of the geodesics on compact closed hyperbolic surface without boundaries (borders), (general case). As specific problems are solved the following tasks: 4) there are studied geodesics on hyperbolic surface of genus g and n (non-puncture) boundary holes (geodesic boundaries); it is given characteristics of all possible types of geodesic launched orthogonally from the point of geodesic boundary of the surface, it is described their behavior and general structure; are studied intervals (horocyclic segments) formed by simple - normal geodesics, launched from the selected conical point, cusp or boundary geodesics on hyperbolic surface. Also, are solved the following problems: 5) a) there are given the characteristics and there are studied properties and types of the geodesics on hyperbolic 1- punctured torus; b) there are studied the geodesics on generalized hyperbolic pants (a sphere with b boundary components and p cusps, with $b+p=3$) and on hyperbolic thrice punctured sphere; c) it is proved that in two dimension the only such manifold not containing a simple closed geodesic is the hyperbolic thrice punctured sphere. But it has six simple complete geodesics.

The results of the preceding paragraphs have allowed solving the problem of the behavior of geodesic in general case: 6) there are described the geodesics for any (oriented) punctured hyperbolic surface M with g genera and k punctures. The proposed new method of the investigation of behavior of the geodesics allowed finally finding the answer of assigned task (behavior of geodesic) and in the most general case: 7) it is solved the question about the qualitative behavior of the geodesics for any hyperbolic surface of signature (g, n, k) (with genus g, k punctures and n geodesic boundaries).

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