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ON RINGS FOR WHICH SOME PRETORSIONS ARE COHEREDITARY

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Abstract: A ring R is characterized by the pretorsions of the category R-Mod of left R-modules: *R* **is completely reducible** if and only if z(R)=0 and every pretorsion r>z is cohereditary, where z is a pretorsion of R-Mod, defined by essential left ideals of R.

Keywords: pretorsion, torsion, cohereditary pretorsion

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Let R be an associative ring with identity and z be a pretorsion, the filter of which consist from the essentially left ideals of a ring R.
Description of rings R over which all pretorsions (or only some of them) possess some properties, presents a considerable interest.

Typical examples:

(1)All pretorsions $r \geq z$ are torsions if and only if R is a left strongly semiprime ring ([4], Theorem, p.80).

(2)All pretorsions $r \geq z$ are superhereditary if and only if R is essentially artinian ring ([5], Theorem, p.110).

In this paper is proved that ring R is nonsingurlar ($z(R)$ =0) and all pretorsions $r > z$ are cohereditary if and only if R is *completely reducible*.

First of all, we present some preliminary notions and definitions.

1. A **preradical r** of R-Mod is a subfunctor of the identity functor of R-Mod ([1-3]).
Every preradical r of R-Mod defines two class of modules:
$R(r) = \{M \in R - Mod \mid r(M) = M\}$ and $P(r) = \{M \in R - Mod \mid r(M) = 0\}$.

Modules of the class $\Re$(r) are called **r-torsion**, and of the class *P*(r) are called r-torsionfree.

Preradicals o and ε for which $P(o) = R - Mod$ and $R(\varepsilon) = R - Mod$ are called zero and identity respectively.

If r and t are preradicals then $r \leq t$ means $r(M) \subseteq t(M)$ for every $M \in R - Mod$.

2. A preradical r is called:
   - a **pretorsion** (or hereditary) if $r(N) = N \cap r(M)$ for any submodule N of an arbitrary module $M \in R - Mod$;
   - **torsion**, if r is a pretorsion and $r\big(M|r(M)\big) = 0$ for every $M \in R - Mod$;
   - **superhereditary** if it is hereditary and the class $\Re$(r) is closed under direct products;
   - **cohereditary** if $r(M|N) = (N + r(M)|N)$ for every $M \in R - Mod$ and every submodule N of M.
3. (a) For any **pretorsion** r and every module M of R-Mod the following equality is true:
$$r(R) = \sum_{\alpha} \{M_\alpha \subseteq M | M_\alpha \in R(r)\}$$
   (b) For any **torsion** r and every module $M \in R - Mod$ the following equality is true:
$$r(M) = \bigcap_{\alpha} \{M_\alpha \subseteq M|\ M|M_\alpha \in P(r)\}$$
   It follows directly from the Proposition 1.5 [1].
4. The **intersection** of pretorsions r_1 and r_2 is the pretorsion $r_1 \wedge r_2$ determined by the rule: $(r_1 \wedge r_2)(M) = r_1(M) \cap r_2(M)$ for any $M \in R - Mod$.
   The **sum** of pretorsion r_1 and r_2 is the preradical $r_1 + r_2$ defined by the relation: $(r_1 + r_2)(M) = r_1(M) + r_2(M)$ for any $M \in R - Mod$.
5. The **Goldie pretorsion z** is a torsion if and only if $z(R) = 0$ ([2], Prop. I.10.2).
6. A ring R is called **strongly semiprime** (SSP), if every essential left ideal P is cofaithful, i.e.
$$(O: P) = \bigcap_{\alpha=1}^{n} (O: p_\alpha) = 0$$
   for some elements $p_\alpha \epsilon P$.
   The following conditions are equivalent:
   (1) R is a left SSP-ring.
   (2) All pretorsion $r \geq z$ are torsions.
   (3) Z(R)=0 and the lattice [z,ε] is complemented.

Passing to the presentation of the basic material we formulate first of all the criterion of **coheredity** of any pretorsion.

Proposition 1. For any pretorsion r the following statements are equivalent:

(1) r is cohereditary;
(2) $r(M) = r(R) \cdot M$ for any $M \in R - Mod$;
(3) r is a torsion and the class *P*(r) is closed under homomorphic images.

Proof. Equivalence of statement (1) and (2) follows from Lemma 3.b [1] or from the Proposition 1.2.8 [2].

Implication (1) ⇒(3) results directly from the definition of the cohereditary pretorsion.

(3) ⇒(1). Let r be a torsion and class *P*(r) is closed under homomorphic images. We will show that for any module M and for any its submodule N the equality $r(M|N) = [N + r(M)]|N$ is true. Indeed, since $[N + r(M)]|N$ is an r-torsion submodule of the module M|N then by the statement 3(a) we have $[N + r(M)]|N \subseteq r(M|N)$. Conversely, since r is a torsion, then by the definition $r(M|[N + r(M)]$ for any module $M \in R - Mod$. But P(r) is closed under homomorphic images the module $M|[N + r(M)]$ is also r-torsion free. Then from the isomorphism $[M|N]|([N + r(M)]|N) \approx M|[N + r(M)]$ we obtain that

$[N + r(M)]|N \subseteq r(M|N) \subseteq [N + r(M)]|N$ imply the equality $r(M|N) = [N + r(M)]|N$. Therefore, by the definitions r is a cohereditary pretorsion. □

In this work, we will show applications of this result.

Proposition 2. For arbitrary pretorsions r and t, the following statements are equivalent:

(1) $R = r(R)\oplus t(R)$ where r and t are choereditary;

(2) $M = r(M)\oplus t(M)$ for any module $M \in R - Mod$.

Proof. (1)⇒(2). From the relation $R = r(R)\oplus t(R)$ we obtain that for any module $M \in R - Mod$ the equality $M = r(M)\oplus t(M)$ is true. Since pretorsions r and t are cohereditary according to the Proposition 1 we have that $r(R) \cdot M = r(M)$ and $t(R)M = t(M)$. Therefore $M = r(M)\oplus t(M)$.

(2)⇒(1). Suppose that for any module $M \in R - Mod$ the equality $M = r(M)\oplus t(M)$ is true. Then, particularly, $R = r(R)\oplus t(R)$. It remains to prove that pretorsions r and t are cohereditary. Indeed, from the equality $M = r(M)\oplus t(M)$ we obtain $M|r(M) \approx t(M)$, so $r\big(M|r(M)\big) = r\big(t(M)\big) = t(M) \cap r(M) = 0$. Therefore, pretosrion r is a torsion. Identically they show that t is a torsion.

Continuing let's show that the classes *P*(r) and *P*(t) are closed under homomorphic images. It is sufficient to verify for the class *P*(r). Let M be an arbitrary r-torsionfree module (r(M)=0). By the assumption, $M = r(M)\oplus t(M) = t(M)$. Then

$r(M|N) = r[t(M)|N] \subseteq t(M)|N \cap r(M|N) \subseteq t(M|N) \cap r(M|N) = 0$ (statement 3(a)).

Therefore, class *P*(r) is closed under homomorphic images. By the Proposition 1 the pretorsion r is cohereditary. Similarly, they show that pretorsion t is also cohereditary. □

Proposition 3. For the pretorsion $r \geq z$ the following statements are equivalent:

(1) The class P(r) is closed under homomorphic images.

(2) Any r-torsionfree module is injective.

(3) Any r-torsionfree module in completely reducible.

Proof. (1)⇒(2). Suppose that class *P*(r) is closed under homomorphic images and M is arbitrary r-torsionfree module. For the injective hull $\widehat{M}$ of a module M we have that $r(\widehat{M}) = 0$ and $r(\widehat{M}|M) = 0$.

From the inclusion $M \subseteq \widehat{M}$ we obtain that $z(\widehat{M}|M) = \widehat{M}|M$, while from inequality $z \leq r$ we obtain $z(\widehat{M}|M) = 0$. But equalities $\widehat{M}|M = z(\widehat{M}|M) = 0$ imply $M = \widehat{M}$. Therefore, M is injective.

(2)⇒(3) is obvious.

(3)⇒(1). Suppose that the module M is r-torsionfree and completely reducible. Then for any its submodule N we have $M = N \oplus C$. Since $r(C) = 0$, therefore the isomorphism $C \cong M|N$ we obtain that $r(M|N) = 0$. Because modules M an N were chosen arbitrarily we have that class *P*(r) is closed under homomorphic images. □

Theoreme 4. The following statements are equivalent:

(1) All pretorsions $r \geq z$ are cohereditary.
(2) $z(R) = 0$ and all pretorsion $r > z$ are cohereditary.
(3) R is a completely reducible ring.
(4) Z is cohereditary.

Proof. Implication (1)⇒(2) is trivial, since z is torsion if $z(R) = 0$. The equivalence of the statement (3) and (4) results directly from Proposition 3 (see Theorem I.10.7 [2]).

(2)⇒(3). Suppose that $z(R) = 0$ and all pretorsion $r > z$ are cohereditary. By the Proposition 1 all pretorsion $r > z$ are torsion. Since the pretorsion z is a torsion ($z(R) = 0$)) we have that all pretorsions $r \geq z$ are torsion, whence we obtain that the ring R is strongly semiprime. By the statement 6 the lattice [z,ε] is complemented. Therefore, for any pretorsion $r > z$ there exists the pretorion $t > z$ such that $z = r \cap t$. Then $z(R) = r(R) \cap t(R) = 0$. Since modules $R|r(R)$ and R|t(R) are respectively r-torsionfree and t-torsionfree (r, t-torsions) and r and t are cohereditary, by the Proposition 3 modules $R|r(R)$ and R|t(R) are completely reducible. The from relation $R = R|[r(R) \cap t(R)] \subseteq R|r(R) \oplus R|t(R)$ it follows that the ring R is completely reducible.

(3)⇒(1). Over any completely reducible ring all pretorsion r are torsion and class *P*(r) is closed under homomorphical images. Bu the Proposition 1 every pretorsion are cohereditary, particularly and all pretorsion $r \geq z$ too. □

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