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ABOUT GLOBAL BEHAVIOUR OF THE GEODESICS ON HYPERBOLIC (LOBACHEVSKY) MANIFOLDS

PhD., Vladimir BALCAN

Academy of Economic Studies of Moldova,
Republic of Moldova, Chisinau, 61 Mitropolitul Banulescu-Bodoni street
Phone: (+373 22) 22 41 28, web site: www.ase.md

Abstract

This paper focuses on the problem of global behavior of straight lines, or „geodesics” on a hyperbolic two-manifold, or surface.

Key words: *behavior of geodesics, hyperbolic pair pants (meaning surfaces of signature (0,3)). hyperbolic surface with genus g, k puncture and n geodesic boundaries.*

JEL CLASSIFICATION: 53 C 22; 57 R 42

The paper is a study of the global behavior of the geodesics on arbitrary two-dimensional hyperbolic manifolds. In this work for the first time systematically is described the geometry of behavior of geodesics on hyperbolic manifolds. We also study the problem related to the existence, number and qualitative properties of geodesics on 2-dimensional hyperbolic manifolds. Even a curved surface can have a notion of a “straight ” line segment : it’s simply the shortest path between two points. On a hyperbolic surface, some geodesics are infinitely long, like straight lines in the plane, but others close up into a loop, like the great circle on a sphere. Geodesics on smooth surfaces are the straightest and locally shortest curves. The generalize the concept of Euclidean straight lines and play a fundamental role in the study of smoothly manifolds. Two basic properties are responsible for their importance: first, that from any point of a manifold there starts a unique geodesic in any direction. Second, the length minimization property (connecting two given points on a manifold with a locally shortest curves). On smooth surfaces geodesics possess both properties. My research is to better understand geodesics on a hyperbolic surface *M*. Much less is known about the behavior of geodesics on hyperbolic surfaces. The chaotic behavior of geodesics on surface of constant negative curvature and finite volume has been known since Hadamard (1898). The problem of understanding the geometry and dynamics of geodesics and rays (i.e. distance-minimizing half geodesics) on hyperbolic manifolds dates back at least to Artin, who started to study the qualitative behavior of geodesics on hyperbolic surfaces. Emil Artin studied the global behavior of geodesics on hyperbolic surfaces by cleverly encoding geodesics using continued fractions.

A major problem we are interested in is to describe of the geodesics trajectories on two-dimensional hyperbolic manifold. We want to understand (describe their) the global behavior of geodesics with a given direction. In particular a) when are geodesics closed ? b) when are the dense in the surface? c) quantitatively, how do they wrap around the surface ? These questions admit notably precise answers, as we are going to see. Geodesics on hyperbolic surfaces are briefly discussed in (see [1, 2, 3]). The following terminology will be used regularly throughout this paper. A (closed) hyperbolic surface can be defined either by a Riemannian metric of constant negative curvature or (thanks to the uniformization theorem) by a quotient of hyperbolic plane by a discrete group of isometries, isomorphic to the fundamental group of the initial surface, acting properly discontinuously on hyperbolic plane. A standard tool in the study of compact hyperbolic surfaces is the decomposition into “pairs of pants” (Y pieces). Given a surface of genus $g \geq 2$, there are $3g-3$ simple closed pairwise non-intersecting geodesics which partition the surface into $g-1$ such

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pieces. A hyperbolic surface of signature (g, n) is an oriented, connected surface of genus g with n boundary components, called boundary geodesics, which is equipped with a metric of constant negative curvature. A hyperbolic surface of genus g with k punctures and n holes and with no boundary is said to be of type (g, k, n). Such surfaces are said to be of finite type. A geodesic in a hyperbolic manifold is a locally distance - minimising curve, and is said to be simple if it has no transverse self-intersections (there-fore it is either an embedded copy of R or an embedded circle) and non simple otherwise. A geodesic on surface M is said to be complete if it is not strictly contained in any other geodesic, i.e., it is either closed and smooth, or open and of infinite length in both directions. Complete geodesics coincide with those which never intersect ∂M. Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp. Throughout, we use the term geodesic to refer to a complete infinite geodesic; a geodesic ray is a half-infinite ray; finally, a geodesic arc is a finite segment lying along some geodesic (which we assume to be closed unless otherwise stated). Recall that the geodesics of the Poincare upper half-plane H model of the hyperbolic plane, are the vertical (half-) lines and the semi-circles centered on the real line. For a hyperbolic surface M some of the geodesics γ will come back to the point they start and fit in a smooth way. These are called closed geodesics. It ends up that there are finitely many closed geodesics of a given lenght (if any).

How do geodesics on the hyperbolic surface behave or how can we determine the behavior of a given geodesic on the hyperbolic surface? The qualitative behavior of geodesics on even seemingly simple hyperbolic surfaces can be surprisingly complex. We investigate in detail the global behavior of geodesics on the simplest hyperbolic surfaces: hyperbolic horn (funnel end), hyperbolic cylinder and parabolic horn (cusp, horn end), or parabolic cylinder. The problem of behavior of geodesic is solvable for a hyperbolic surface called hyperbolic horn. A hyperbolic horn is a two-dimensional manifold , obtained from the strip between the two parallel straight lines of the hyperbolic plane by matching the border lines by shifting (sliding), its axis being parallel to the border lines and beyond the strip between them. The hyperbolic horn, i.e. the factor-space $H^2_+\backslash\Gamma$, is an (open) half of the hyperbolic cylinder. The border circumference does not belong to that half and there for the surface of the hyperbolic horn is incomplete. The funnel is half of the hyperbolic cylinder, bounded by their closed geodesic. The full funnel continues to flare out exponentially and has infinite area. There is a

Theorem 1. On the hyperbolic horn the problem of behavior of a geodesic is solvable. The theorem is resolved using the affirmations I-IV set out below.

Affirmation I. There are no closed geodesics on the hyperbolic horn.

Affirmation II. If the geodesic *l* on the hyperbolic horn $M^2=H^2_+/\Gamma$ is defined so that its covering (a lift) lies on a straight line intersecting the line *a*, then the geodesic l is infinite without self-intersections and any of its points divides it into two rays: one ray of finite length, another ray of infinite length.

Affirmation III. If the covering (a lift) *l’* for the geodesic *l* for the hyperbolic horn M^2 is the straight line parallel to the line *a*, then the geodesic l is infinite, without self-intersections points, and any of its points divides it into two congruent rays.

Affirmation IV. If the covering *l’* for the geodesic *l* is a straight line divergent with the axis of shifts, then the geodesic *l* is infinite and it has only a finite number k of double self-intersection points.

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Here in none of the cases the geodesic was not a closed one, as said in the Affirmation I. Therefore, in each of the three possible cases the behavior of geodesic is fully described, and since any other cases are impossible, it has been demonstrated that the behavior of geodesic on hyperbolic horn is fully solvable. So, every geodesic curve γ on the hyperbolic horn is of one of the four types: 1) there are no closed geodesics; 2) there is a geodesic of infinite length, without self-intersections points, and any of its points divides the geodesic into two rays: one ray of finite length and another ray of infinite length; 3) there is an infinite geodesic, without self-intersections points and any of its points divides it into two congruent rays; 4) there is an infinite geodesic and it has a finite number *k* of double self-intersection points and they are all divisible by 2. The number *k* of self-intersection points of an examined geodesic is equal to *p.* The problem of behavior of a geodesic on a hyperbolic cylinder is solvable. One may define the hyperbolic cylinder as a non-compact two-dimensional manifold obtained from the strip from between the two divergent lines of the hyperbolic plane by identifying the divergent border lines by shift (sliding), its axis being a common perpendicular for the said border lines, its shift being equal to the length of such translation. The factor space $H^2\backslash\Gamma$ is a some kind of cylindrical surface also called hyperbolic cylinder. The hyperbolic cylinder is the union of two funnels.

Theorem 2. On the hyperbolic cylinder $C = H^2/\Gamma$ the geodesic's behavior problem is solvable.

The proof of theorem comes from the affirmations I and II set out below.

Affirmation I. There are no closed geodesics on the cylinder *C* (both simple, different from the narrow geodesic core of cylinder and non-simple ones).

This results from the fact that the closed geodesics $\tilde{b}$ correspond to the translation $\vec{b}$. But such translation should transform into itself the straight line *a*, while this is possible only when the line *b* is on the line *a*, i.e. it is a translation along the line *a.* This translation along the line *a* on a hyperbolic cylinder will lie on a geodesic core (the narrowest place of cylinder). It is the only simple closed geodesic on that surface.

*Affirmation II***.** If the geodesic's image intersects the straight line *a,* such a geodesic is a geodesic without self-intersection points, infinite in both directions (at both ends).

Let us consider the behavior of geodesic on a parabolic cusp (parabolic cylinder). We shall call a parabolic horn (cusp) the two-dimensional manifold obtained from the strip from between the two parallel lines of the hyperbolic plane by identifying the border lines by horocyclic rotation determined by these lines. The parabolic cylinder is a special case (its small end is a cusp, while the “horn” end carriers the hyperbolic metric). There appears the

Theorem 3. The problem of behavior of a geodesic on a horn end (cusp) is solvable.

The study of universal cover of parabolic cusp demonstrates that:

1. if the arbitrary straight line *c* does not cross the obstructing line of the pair determining the horocyclic rotation *w* and identified upon that rotation, the image of the said straight line on this surface(cusp) is isometric to the usual straight line of a hyperbolic surface *(*simple infinite length, without self-intersection*)*;

2. if the image of the geodesic *c* on the hyperbolic plane H^2 is a straight line intersecting the said geodesic and if it is different from the obstructing straight line, then the geodesic c is infinite in both directions (at both ends) and it has only a finite number k of double self-intersection points. In the particular case, both ends of the geodesic can go to the some point at infinity.

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3. there are no closed geodesics on the parabolic cusp, because no translation in the group $\Gamma = \langle w \rangle$.

The study of geodesics on hyperbolic surfaces can be reduced to the study of curves on a hyperbolic pair of pants. Compact hyperbolic surfaces can be seen as an elementary pasting of geodesic polygons of the hyperbolic plane. Conversely, cutting such a surface along disjoint simple closed geodesics (a partition), one obtains a family of pair of pants (surfaces of signature (0,3)), which in turn can be readily cut to obtain a pair of isometric right-angled hexagons. Let M be a surface and let P be a pair of pants. In this paper, we focus on getting the behavior of geodesics on a hyperbolic pair of pants P. As a direct consequence we get the behavior of geodesics on any surface M. We do this as follows. First, there is a unique way to write P as the union of two congruent right-angled hexagons. Take this decomposition (see on Fig.1).We examining different types of behaviors exhibited by geodesics on a given pair of hyperbolic pants and study infinite simple geodesic rays and complete geodesics. We also allow the degenerate case in which one or more of the lenths vanish (a generalized pair of pants). We call a generalized pair of pants a hyperbolic surface which is a homeomorphic to a sphere with three holes, a hole being either a geodesic boundary component or a cusp. Main results of the present work are as follows. In the work is given a constructive method for solving the problem of the behavior of geodesic on a arbitrary hyperbolic surfaces of signature (g, n, k), i.e., method allowing to answer the question about the structure on the global of examined geodesic at its indefinitely extension on both directions. Such a compressed formulated result can be disclosed as follows. For this purpose, with the help of proposed practical approach at first are studied geodesics at the simplest hyperbolic manifolds.: 1) it is solved the problem of the behavior of geodesic on the simplest hyperbolic surfaces (hyperbolic horn; hyperbolic cylinder; parabolic cusp); 2) it is investigated and described the behavior of geodesic lines on hyperbolic surfaces of signature (0,3) (hyperbolic pants); it is found special case: behavior of ortho-boundary geodesics and orthogeodesics and their general structure, i.e., it is obtained classification of geodesics emanating normally from the point of geodesic boundary of pants(see on Fig.2).

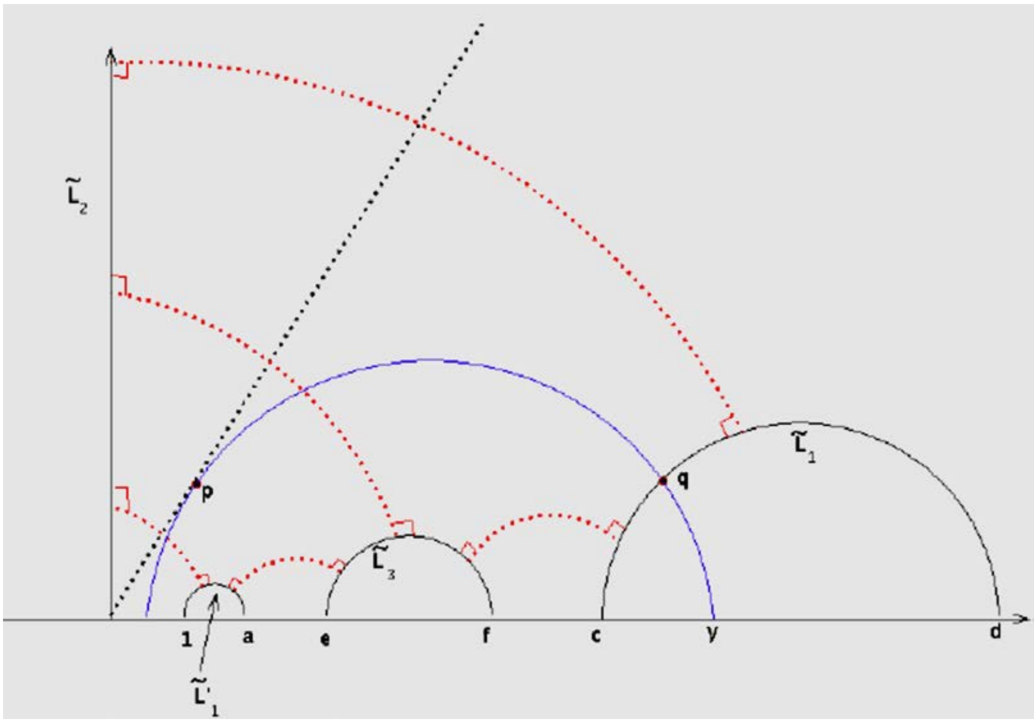

Figure 1. Universal cover of *P* in H^2

Investigation of behavior of geodesics on the listed above surfaces, allowed finding answer of assigned task in the most general case: 3) it is investigated and found behavior of geodesics on compact closed hyperbolic surface without boundaries, (general case). As specific problems are solved the following tasks: 4) there are studied geodesics on hyperbolic surface of genus g and n geodesic boundaries; it is given characteristics of all possible types of geodesic launched orthogonally from the point of geodesic boundary of the surface, it is described their behavior and general structure; are studied intervals formed by simple - normal geodesics, launched from the selected conical point, cusp or boundary geodesics on hyperbolic surface. Also, are solved the following problems: 5) a) there are given the characteristics and there are studied properties and types of geodesics on hyperbolic 1- punctured torus; b) there are studied geodesics on generalized hyperbolic pants (a sphere with b boundary components and p cusps, with $b+p=3$) and on hyperbolic thrice punctured sphere; c) it is proved that in two dimension the only such manifold not containing a simple closed geodesic is the hyperbolic thrice punctured sphere. But it has six simple complete geodesics.

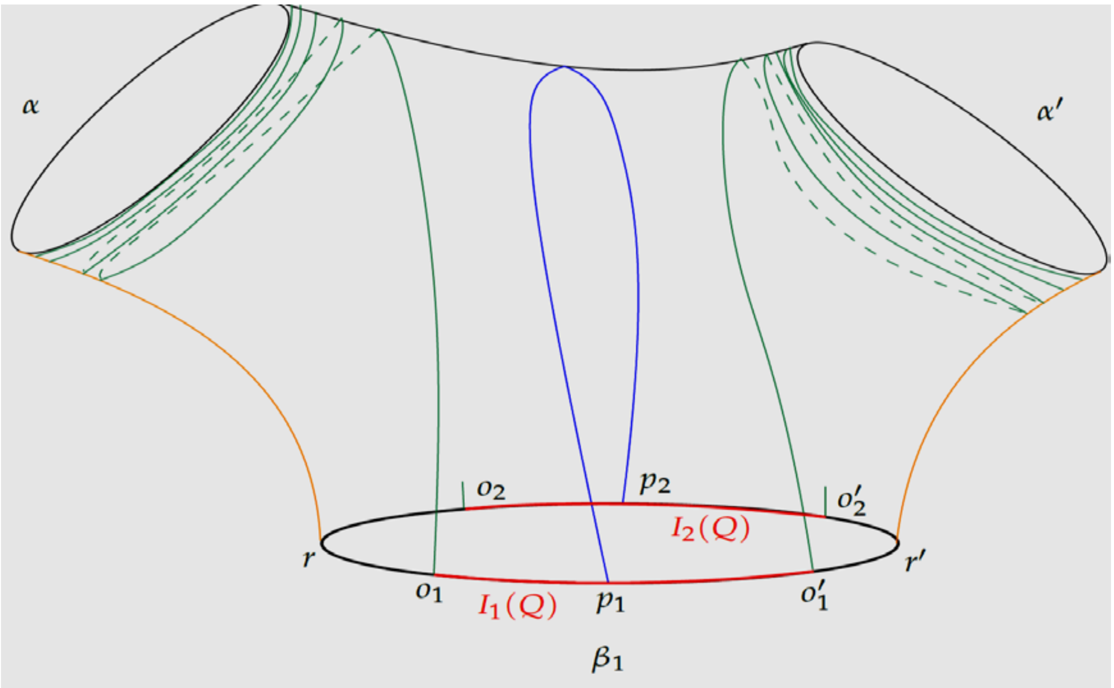

Figure 2. Pants *P* with spiraling geodesics

The results of the preceding paragraphs have allowed solving the problem of the behavior of geodesic in general case: 6) there are described geodesics for any (oriented) punctured hyperbolic surface M with genus g and k punctures. The proposed new method of the investigation of behavior of geodesics allowed finally finding the answer of assigned task (behavior of geodesic) and in the most general case: 7) it is solved the question about the qualitative behavior of geodesics for any hyperbolic surface of signature (g, n, k) (with genus g, k punctures and n geodesic boundaries).

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