SESSION VII: INFORMATION TECHNOLOGIES AND ECONOMIC CYBERNETICS

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GENERATING OF HAMILTON FULL FAVORING APPORTIONMENTS

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Abstract. *Aspects of full favoring of small beneficiaries in apportionments using Hamilton method are discussed. For this purpose, the requirements of full favoring Hamilton apportionments were defined and the A_{HS} algorithm for determining such apportionments is described. Then, calculations which confirm the opportunity of using the A_{HS} algorithm in this aim were performed.*
Key words: *algorithm, apportionment problem, favoring of small beneficiaries, Hamilton method*

JEL CLASSIFICATION: C61, C63

1. Introduction

Often it is necessary to distribute a given number M of discrete entities of the same kind among n beneficiaries, in proportion to a numerical characteristic assigned to each of them V_i, $i = \overline{1,n}$. This is known as proportional apportionment (APP) problem (Balinski & Young, 2001; Kohler & Zeh, 2012; Niemeyer & Niemeyer, 2008). The integer character of this problem usually causes a certain disproportion of the apportionment $\{x_i, i = \overline{1,n}\}$ (Balinski & Young, 2001; Gallagher, 1991; Karpov, 2008), some beneficiaries being favored at the expense of the others. Such favoring leads to the increase of disproportionality of the apportionment. Therefore, reducing the favoring in question is one of the basic requirements when is choosing the APP method to be applied for apportionments.

As it is well known, the d'Hondt method (d'Hond, 1878) favors large beneficiaries (with larger V_i value) (Gallagher, 1991; Sorescu et al., 2006; Bolun, 2016), and Huntington-Hill method (Huntington, 1921) favors the small ones (with smaller V_i value) (Gallagher, 1991; Tannenbaum, 2008). But which of the two favors beneficiaries to a larger extent? Preferences, in this sense, between methods, can help. Par example, in (Marshall et al., 2002), five APP methods are placed „in the order as they are known to favor larger parties over smaller parties". However, the best way is to estimate this property quantitatively. One approach in this aim is proposed in (Bolun, 2020). Another, the "total (full) favoring", based on the definition of favoring of large beneficiaries or of the small ones by an APP method done in (Balinski & Young, 2001), is examined in (Bolun, 2020). In (Bolun, 2020), it was shown that the frequency of full favoring in apportionments, for the widely used Hamilton (Hare) (Hare, 1859), Sainte-Laguë (Webster) (Sainte-Laguë, 1910), d'Hondt (Jefferson), Huntington-Hill and Adapted Sainte-Laguë methods, is strongly decreasing on n, becoming approx. 0 at $n \geq 7 \div 10$. Aspects of the guaranteed generation of Hamilton apportionments, which fully favor small beneficiaries at larger values of n, are examined in this paper.

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2. Essence of favoring of beneficiaries in apportionments

The essence of favoring of beneficiaries in apportionments is described in such papers as (Gallagher, 1991; Sorescu et al., 2006]. From the three notions of favoring of beneficiaries by an APP method distinguished in (Bolun, 2020), the following two will be used in this paper:

a) favoring of a beneficiary in an apportionment;

b) favoring of small beneficiaries in an apportionment.

It is considered that a beneficiary i is favored if a larger number x_i of entities is distributed to him than would be due according to the V_i value, quantitatively if $x_i > MV_i /V$, where $M = x_1 + x_2 + \ldots + x_n$ and $V = V_1 + V_2 + \ldots + V_n$. Of course, the lack of favoring is possible only if the equalities $a_i = MV_i/V$, $i = \overline{1,n}$ take place; here $a_i = \lfloor MV_i/V \rfloor$, where $\lfloor z \rfloor$ means the integer part of the real number z. In practice, such equalities rarely occur and that is why some beneficiaries are favored and others, respectively, are disfavored. The notation $\Delta M = M - (a_1 + a_2 + \ldots + a_n)$ will also be used.

In formalized form, the first, probably, definition of favoring of large beneficiaries or of the small ones by an APP method is given in (Balinski & Young, 2001). But the requirements of this definition are very strong - no method compliant to them and used in practice is known. At the same time, as mentioned in (Bolun, 2021), these conditions can be used to identify the "full favoring" of large beneficiaries or of the small ones in particular apportionments. Also, in (Bolun, 2020), the requirements of the respective definition in (Balinski & Young, 2001) were simplified, reducing considerably the volume of needed calculations for computer simulation (see Definition 1).

Definition 1. In an apportionment, small beneficiaries are fully favored if

$$\frac{x_i}{V_i} < \frac{x_j}{V_j}, \qquad (1)$$

whenever $x_i > x_j$, where $(i, j) \in \{1, 2, 3, \ldots, n\}$ (Bolun, 2020).

Usually, in one and the same apportionment some large and some small beneficiaries can be favored and, nevertheless, mainly large or, on the contrary, mainly small beneficiaries can be favored. Therefore, in (Bolun, 2020) it is proposed to use two different notions: "favoring" of large or of small beneficiaries and "full favoring" of large or of small beneficiaries, the second being a particular case of the first one. The compliance of an apportionment with requirement (1) is referred to "full favoring" of small beneficiaries. The larger notion of „favoring" is used when in an apportionment are predominantly favored large beneficiaries or, on the contrary, the small ones in sense of (Bolun, 2020).

In order to identify whether apportionments that fully favor small beneficiaries can be obtained when applying the Hamilton APP method, it is necessary to know the compliance conditions of this method with requirements (1).

3. Compliance of an apportionment with the Hamilton solution

The required apportionments must be Hamilton and, at the same time, comply with requirements (1). The conditions for the compliance of an apportionment with the solution obtained by Hamilton method (Hamilton apportionment) are defined by Statement 1. First, let: $Q = V/M$; $V_i = a_iQ + \Delta V_i > 0$, $i = \overline{1,n}$; $\Delta M = (\Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots + \Delta V_n)/Q$, $1 \le l \le n-1$ and $x_i > x_{i+1}$, $i = \overline{1, n-1}$. Of course, occur $0 \le \Delta V_i < Q$, $i = \overline{1,n}$.

Statement 1. The necessary conditions for the compliance of an apportionment $\{x_i, i = \overline{1,n}\}$, which fully favors small beneficiaries, with the solution obtained by Hamilton method are

$$\Delta V_i < \Delta V_k, i = \overline{1, n-l}, k = \overline{n-l+1, n}. \qquad (2)$$

Indeed, the Hamilton method apportionment rule states (Gallagher, 1991; Tannenbaum, 2008) that in addition to the already apportioned a_i entities, $i = \overline{1,n}$, the remained unapportioned $\Delta M = l$ entities should be apportioned by one to the first beneficiaries with the largest ΔV_j value. So, taking into account that $x_i > x_{i+1}$, $i = \overline{1,n-1}$, the relations $x_i = a_i$, $i = \overline{1,n-l}$ and $x_i = a_i + 1$, $i = \overline{n-l+1,n}$ should take place when favoring small beneficiaries that can be only if occurs (2). ■

It should be mentioned that Statement 1 establishes relationships between beneficiaries of two groups, $\{i = \overline{1,n-l}$ and $i = \overline{n-l+1,n}\}$, but not between beneficiaries within each of these groups if $n > 2$, needed to establish when analyzing the full favoring of small beneficiaries according to requirements (1).

4. Compliance of Hamilton apportionments with requirements (1)

It is well known that overall, on an infinity of apportionments, Hamilton method doesn't favor beneficiaries (Balinski & Young, 2001; Gallagher, 1991; Tannenbaum, 2008). But it can be particular Hamilton apportionments which favor small beneficiaries. Moreover, as confirmed below, some of such apportionments fully favor small beneficiaries. The respective conditions are defined by Statement 2.

Statement 2. If $n > 2$ and $l = \Delta M$, the conditions for the compliance of a Hamilton apportionment $\{x_i, i = \overline{1,n}\}$ with the requirement (1) of full favoring of small beneficiaries, in addition to the (2) ones, are

$$\Delta V_i > \frac{a_i}{a_{i+1}} \Delta V_{i+1}, i = \overline{1,n-l-1} \tag{3}$$

if $1 = l < n - 1$ (Case S1),

$$\Delta V_{i+1} < \frac{\Delta V_i(a_{i+1} + 1) + Q(a_i - a_{i+1})}{a_i + 1}, i = \overline{n-l+1,n-1} \tag{4}$$

if $1 < l = n - 1$ (Case S2) and both, (3) and (4), if $1 < l < n - 1$ (Case S3).

Indeed, one has $0 \le \Delta V_i < Q$, $i = \overline{1,n}$ and, because of $1 = l < n - 1$, in (3) the relations $a_{i+1} > 0$, $i = \overline{1,n-l-1}$ always occurs. Let's begin with **Case S3**, divided into the following three subcases:

S3a) $x_i = a_i$, $x_k = a_k$, $i = \overline{1,n-l-1}$, $k = \overline{i+1,n-l}$;

S3b) $x_i = a_i$, $x_k = a_k + 1$, $i = \overline{1,n-l}$, $k = \overline{n-l+1,n}$;

S3c) $x_i = a_i + 1$, $x_k = a_k + 1$, $i = \overline{n-l+1,n-1}$, $k = \overline{i+1,n}$.

In **Subcase S3a**, according to (1) it should be

$$\frac{x_i}{V_i} < \frac{x_k}{V_k}, \text{that is } \frac{a_i}{a_i Q + \Delta V_i} < \frac{a_k}{a_k Q + \Delta V_k}, i = \overline{1,n-l-1}, k = \overline{i+1,n-l},$$

from where one has

$$\Delta V_k < \frac{a_k}{a_i} \Delta V_i, i = \overline{1,n-l-1}, k = \overline{i+1,n-l}. \tag{5}$$

It is easy to show that requirements (5) are transitive. From (5), one has

$$\Delta V_i > \frac{a_i}{a_{i+1}} \Delta V_{i+1} \text{ and } \Delta V_{i+1} > \frac{a_{i+1}}{a_{i+2}} \Delta V_{i+2}, \text{from where } \Delta V_i > \frac{a_i}{a_{i+1}} \frac{a_{i+1}}{a_{i+2}} \Delta V_{i+2} = \frac{a_i}{a_{i+2}} \Delta V_{i+2}.$$

In the same way one can show that relations $\Delta V_i > \frac{a_i}{a_{i+j}} \Delta V_{i+j}, i = \overline{1,n-l-1}, k = \overline{i+1,n-l}$ occur. Thus, relations (5) are transitive and can be replaced by the (3) ones. ▼

In **Subcase S3b**, according to (1) it should be

$$\frac{x_i}{V_i} < \frac{x_k}{V_k}, \text{that is } \frac{a_i}{a_iQ + \Delta V_i} < \frac{a_k + 1}{a_kQ + \Delta V_k}, i = \overline{1, n - l}, k = \overline{n - l + 1, n},$$

from where one has $a_i\,(\Delta V_k - Q) < \Delta V_i\,(a_k + 1)$. Because of $0 \leq \Delta V_k < Q$ and $\Delta V_i\,(a_k + 1) \geq 0$, the requirements $a_i\,(\Delta V_k - Q) < \Delta V_i\,(a_k + 1)$, $i = \overline{1, n - l}, k = \overline{n - l + 1, n}$ always take place, that's why Subcase S2b is not specified in Statement 2. ▼

In **Subcase S3c**, according to (1) it should be

$$\frac{x_i}{V_i} < \frac{x_k}{V_k}, \text{that is } \frac{a_i + 1}{a_iQ + \Delta V_i} < \frac{a_k + 1}{a_kQ + \Delta V_k}, i = \overline{n - l + 1, n - 1}, k = \overline{\iota + 1, n},$$

from where one has

$$\Delta V_k < \frac{\Delta V_i(a_k + 1) + Q(a_i - a_k)}{a_i + 1}, i = \overline{n - l + 1, n - 1}, k = \overline{\iota + 1, n}. \quad (6)$$

Let's show that requirements (6) are transitive. From (6), for $k = i + 1$ one has

$$\Delta V_i > \frac{\Delta V_{i+1}(a_i + 1) - Q(a_i - a_{i+1})}{a_{i+1} + 1}, i = \overline{n - l + 1, n - 1} \quad (7)$$

and, respectively,

$$\Delta V_{i+1} > \frac{\Delta V_{i+2}(a_{i+1} + 1) - Q(a_{i+1} - a_{i+2})}{a_{i+2} + 1}, i = \overline{n - l + 1, n - 2}. \quad (8)$$

Taking into account (8), requirement (7) can be transformed as follow

$$\Delta V_i > \frac{1}{a_{i+1} + 1}\left[(a_i + 1)\frac{\Delta V_{i+2}(a_{i+1} + 1) - Q(a_{i+1} - a_{i+2})}{a_{i+2} + 1} - Q(a_i - a_{i+1})\right] =$$

$$= \frac{\Delta V_{i+2}(a_i + 1) - \frac{Q(a_{i+1} - a_{i+2})}{a_{i+1} + 1}(a_i + 1) - \frac{Q(a_i - a_{i+1})}{a_{i+1} + 1}(a_{i+2} + 1)}{a_{i+2} + 1} =$$

$$= \frac{\Delta V_{i+2}(a_i + 1) - Q(a_i - a_{i+2})}{a_{i+2} + 1}, i = \overline{n - l + 1, n - 2}. \quad (9)$$

So, if relations (7) and (8) take place, than relation (9) occurs, too. The same way, one can show that occurs

$$\Delta V_i > \frac{\Delta V_{i+j}(a_i + 1) - Q(a_i - a_{i+j})}{a_{i+j} + 1}, i = \overline{n - l + 1, n - 1}, j = \overline{1, n - \iota}. \quad (10)$$

Thus, requirements (6) are transitive and therefore they can be replaced by the (4) ones. ▼

The proof for **Cases S1** and **S2**, taking into account proves for Subcases S3a and S3c, are trivial. ■

When generating apportionments which fully favor small beneficiaries, the inequalities

$$\Delta V_i < \frac{a_i}{a_{i-1}}\Delta V_{i-1}, i = \overline{2, n - l}, \quad (11)$$

$$\Delta V_i < \frac{\Delta V_{i-1}(a_i + 1) + Q(a_{i-1} - a_i)}{a_{i-1} + 1}, i = \overline{n - l + 2, n}, \quad (12)$$

equivalent to the (3) and (4) ones, are also useful.

5. Generating Hamilton apportionments that fully favor small beneficiaries

Based on Statements 1 and 3, the **A$_{HS}$ algorithm** for the generation of Hamilton apportionments that fully favor small beneficiaries was elaborated. According to (3), the lower the value of ΔV_{n-l}, the lower the values of ΔV_i, $i = \overline{1, n-l-1}$. Similarly, according to (4), the lower the value of ΔV_n, the lower the values of ΔV_i, $i = \overline{n-l+1, n-1}$. Taking into account these observations and considering $V > M$ and that the value of ΔM is known, in Figure 1 the basic conceptual steps of the A_{HS} algorithm are shown.

At Steps 3 and 4 of the A_{HS} algorithm, minimal possible values to $\Delta V_i \geq 0$, $i = \overline{1, n}$ are allocated: at Step 3 - to $\Delta V_i \geq 0$, $i = \overline{1, n-l}$ according to requirement (3) and beginning with the value of $\Delta V_{n-l} > 0$; at Step 4 - to $\Delta V_i \geq 0$, $i = \overline{n-l+1, n}$ according to requirement (4) and beginning with the value of $\Delta V_n > z = \max\{\Delta V_1, \Delta V_2, \Delta V_3, \ldots, \Delta V_{n-l}\}$ because of requirement (2). If after these allocations one has $\Delta M > l$, that is $\Delta V > \Delta U$, then the solution doesn't exist.

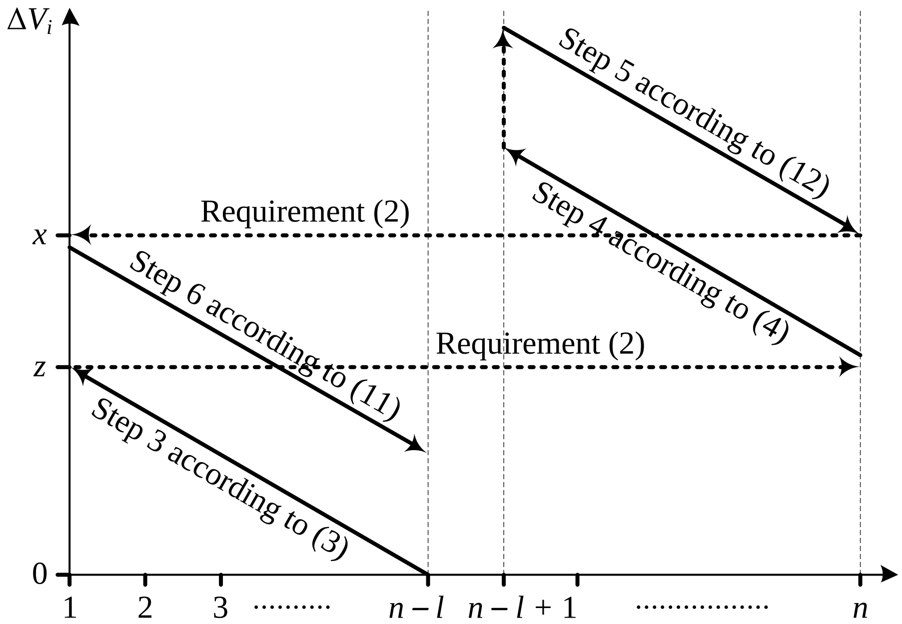

Figure 1. Basic steps of the A_{HS} Algorithm.
Source: elaborated by the author.

On the contrary, if $\Delta M < l$, that is if $\Delta V < \Delta U$, then one has to increase ΔV aiming to reach $\Delta V = \Delta U$. Because of requirement (4), it is relevant to increase first, maximal possible, the values of ΔV_i, $i = \overline{n-l+1, n}$ beginning with $\Delta V_{n-l+1} < Q$. This is done at Step 5 according to requirement (12). But if at this step the equality $\Delta V = \Delta U$ is not achieved, then the last possibility to increase the value of ΔV is the increase of ΔV_i, $i = \overline{1, n-l}$ values beginning with $\Delta V_1 < x = \min\{\Delta V_i, i = \overline{n-l+1, n}\}$ because of requirement (4). This is done at Step 6 according to requirement (11).

It should be mentioned that in Figure 1 a continuous arrow doesn't reflect the relation between the values of ΔV_i and ΔV_{i-1} sizes. It reflects the relation between ΔV_i and the respective function of:

1) ΔV_{i+1} (at Steps 3 and 4), that is $\Delta V_i > f_3(\Delta V_{i+1})$ according to requirement (3) and, respectively, the (4) one;
2) ΔV_{i-1} (at Steps 5 and 6), that is $\Delta V_i < f_4(\Delta V_{i-1})$ according to requirement (11) and, respectively, the (12) one.

The A_{HS} algorithm in details is the following.

1. Initial data are: V, n, $1 \leq l \leq n-1$, $1 \leq g \leq \lceil Q/n \rceil$ and $x_i > x_{i+1}$, $i = \overline{1, n-1}$.
2. $M := x_1 + x_2 + x_3 + \ldots + x_n$, $Q := V/M$, $\Delta U := Ql$; $a_i = x_i$, $i = \overline{1, n-l}$; $a_i = x_i - 1$, $i = \overline{n-l+1, n}$.

3. Based on (3), determining the preliminary, minimal possible, values of sizes $\Delta V_i \geq 0$, $i = \overline{1, n-l}$.
   3.1. $i := n - l$. $\Delta V_i := \lfloor Qa_i \rfloor + 1 - Qa_i$. If $i = 1$, then go to Step 4.
   3.2. $i := i - 1$. $\Delta V_i := \lfloor Qa_i + \Delta V_{i+1}\, a_i / a_{i+1} \rfloor + g - Qa_i$. If $\Delta V_i \geq Q$, then the solution doesn't exist. Stop.
   3.3. If $i > 1$, then go to Step 3.2.
4. Based on (4), determining the preliminary, minimal possible, values of sizes $\Delta V_i > 0$, $i = \overline{n-l+1, n}$.
   4.1. $z := \max\{\Delta V_1, \Delta V_2, \Delta V_3, \ldots, \Delta V_{n-l}\}$; $\Delta V := \Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots + \Delta V_{n-l}$.
   4.2. $i := n$. $\Delta V_i := \lfloor Qa_i + z \rfloor + g - Qa_i$. If $\Delta V_i \geq Q$, then the solution doesn't exist. Stop.
   4.3. If $l = 1$, then go to Step 5.
   4.4. $i := i - 1$. $\Delta V_i := \lfloor Qa_i + [\Delta V_{i+1}(a_i + 1) - Q(a_i - a_{i+1})]/(a_{i+1} + 1) \rfloor + g - Qa_i$. If $\Delta V_i \geq Q$, then the solution doesn't exist. Stop.
   4.5. If $\Delta V_i \leq z$, then it is needed to minimally increase ΔV_i. $\Delta V_i := \lfloor Qa_i + z \rfloor + g - Qa_i$. If $\Delta V_i \geq Q$, then the solution doesn't exist. Stop.
   4.6. If $i > n - l + 1$, then go to Step 4.4.
5. Based on (12), ensuring $\Delta M = l$ by maximal possible increasing, if needed, the $\Delta V_i > 0$, $i = \overline{n-l+1, n}$ values.
   5.1. $\Delta V := \Delta V + \Delta V_{n-l+1} + \Delta V_{n-l+2} + \Delta V_{n-l+3} + \ldots + \Delta V_n$. If $\Delta V > \Delta U$, then the solution doesn't exist. Stop.
   5.2. If $\Delta V = \Delta U$, then the solution is obtained. Go to Step 7.
   5.3. $y := \Delta U - \Delta V$, $i := n - l + 1$. If $Q - \Delta V_i > y$, then $\Delta V_i := \Delta V_i + y$ and the solution is obtained. Go to Step 7.
   5.4. $h := \Delta V_i$, $\Delta V_i := \lceil Qa_i + Q \rceil - g - Qa_i$, $y := y - \Delta V_i + h$. If $l = 1$, then it is needed to increase the values of ΔV_i, $i = \overline{1, n-l}$. Go to Step 6.
   5.5. $i := i + 1$; $h := \Delta V_i$; $\Delta V_i := \lceil Qa_i + [\Delta V_{i-1}(a_i + 1) + Q(a_{i-1} - a_i)]/(a_{i-1} + 1) \rceil - g - Qa_i$. If $\Delta V_i < Q$, then:
      5.5.1. If $\Delta V_i > h + y$, then $\Delta V_i := h + y$ and the solution is obtained. Go to Step 7.
      5.5.2. $y := y - \Delta V_i + h$ and go to Step 5.8.
   5.6. If $Q > h + y$, then $\Delta V_i := h + y$ and the solution is obtained. Go to Step 7.
   5.7. $\Delta V_i := \lceil Qa_i + Q \rceil - g - Qa_i$; $y := y - \Delta V_i + h$.
   5.8. If $i < n$, go to Step 5.5.
6. Based on (11), ensuring $\Delta M = l$ by the maximal possible increase of the $\Delta V_i \geq 0$, $i = \overline{1, n-l}$ values.
   6.1. $x := \min\{\Delta V_i, i = \overline{n-l+1, n}\}$. $i := 1$, $h := \Delta V_i$. If $x > h + y$, then $\Delta V_i := h + y$ and the solution is obtained. Go to Step 7.
   6.2. $\Delta V_i := \lceil Qa_i + x \rceil - g - Qa_i$. $y := y - \Delta V_i + h$.
   6.3. If $i = n - l$, then the solution doesn't exist. Stop.
   6.4. $i := i + 1$, $h := \Delta V_i$. $\Delta V_i := \min\{\lceil Qa_i + x \rceil; \lceil Qa_i + \Delta V_{i-1}\, a_i / a_{i-1} \rceil\} - g - Qa_i$. If $\Delta V_i > h + y$, then $\Delta V_i := h + y$ and the solution is obtained. Go to Step 7.
   6.5. $y := y - \Delta V_i + h$ and go to Step 6.3.
7. Determining the V_i, $i = \overline{1, n}$ values. $V_i := Qa_i + \Delta V_i$, $i = \overline{1, n}$. Stop.

The obtained V_i, $i = \overline{1, n}$ values can be checked by applying the Hamilton method. To note, that the affirmations "the solution doesn't exist" in the A_{HS} algorithm are approximate, but very close to reality for $g = 1$. Parameter g is an integer, which value influences the minimal difference among the $x_{i+1}/V_{i+1} - x_i/V_i$, $i = \overline{1, n-1}$ ones: the larger the value of g, the larger the mentioned difference. At the same time, the smaller the value of g, the higher the probability that the solution will be obtain.

Algorithm A_{HS} was implemented in the computer application SIMAP. Examples 1, 2, 3 and 4 using SIMAP are described below.

Example 1 regarding the generation of a Hamilton apportionment which fully favors small beneficiaries. Initial data: M = 279; n = 20; ΔM = 10; V = 20000; g = 1; the x_i, $i = \overline{1,n}$ values specified in Table 1. Some results of calculations using SIMAP are systemized in Table 1.

Table 1. Calculations for the apportionment to Example 1

i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$
1	2156	30	139147	6	1436	20	139276	11	931	13	139635	16	427	6	140515
2	1940	27	139175	7	1364	19	139296	12	787	11	139771	17	284	4	140845
3	1796	25	139198	8	1292	18	139319	13	715	10	139860	18	212	3	141509
4	1652	23	139225	9	1148	16	139373	14	571	8	140105	19	141	2	141844
5	1580	22	139241	10	1004	14	139442	15	499	7	140281	20	65	1	153846

Source: elaborated by the author.

Example 2 regarding the generation of a Hamilton apportionment which fully favors small beneficiaries. Initial data are the same as in Example 1 with the only difference that g = 2. Some results of calculations using SIMAP are systemized in Table 2.

Table 2. Calculations for the apportionment to Example 2

i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$
1	2170	30	138249	6	1440	20	138889	11	930	13	139785	16	423	6	141844
2	1952	27	138320	7	1367	19	138991	12	785	11	140127	17	280	4	142857
3	1806	25	138428	8	1294	18	139104	13	712	10	140449	18	208	3	144231
4	1660	23	138554	9	1149	16	139252	14	568	8	140845	19	137	2	145985
5	1586	22	138714	10	1004	14	139442	15	495	7	141414	20	34	1	294118

Source: elaborated by the author.

Example 3 regarding the generation of a Hamilton apportionment which fully favors small beneficiaries. Initial data are the same as in Example 1 with the only difference that g = 3. Some results of calculations using SIMAP are systemized in Table 3.

Table 3. Calculations for the apportionment to Example 3

i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$
1	2185	30	137300	6	1446	20	138313	11	929	13	139935	16	419	6	143198
2	1964	27	137475	7	1371	19	138585	12	784	11	140306	17	277	4	144404
3	1816	25	137665	8	1296	18	138889	13	710	10	140845	18	185	3	162162
4	1668	23	137890	9	1150	16	139130	14	565	8	141593	19	109	2	183486
5	1593	22	138104	10	1004	14	139442	15	492	7	142276	20	37	1	270270

Source: elaborated by the author.

Example 4 regarding the generation of a Hamilton apportionment which fully favors small beneficiaries. Initial data are the same as in Example 1 with the only difference that g = 4. Some results of calculations using SIMAP are systemized in Table 4.

Table 4. Calculations for the apportionment to Example 4

i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$	i	V_i	x_i	$10^{-7}x_i/V_i$
1	2196	30	136612	6	1450	20	137931	11	928	13	140086	16	407	6	147420

2	1973	27	136847		7	1374	19	138282		12	776	11	141753		17	264	4	151515
3	1824	25	137061		8	1298	18	138675		13	691	10	144718		18	192	3	156250
4	1675	23	137313		9	1151	16	139010		14	550	8	145455		19	121	2	165289
5	1599	22	137586		10	1004	14	139442		15	478	7	146444		20	49	1	204082

Source: elaborated by the author.

Data of Tables 1-4 were checked – the obtained apportionments are Hamilton ones. At the same time, they comply with requirements (1). Thus, they fully favor small beneficiaries.

Comparing data in Tables 1, 2, 3 and 4, one can see that the obtained values of V_i and x_i/V_i, $i = \overline{1,n}$ differ. Using different values of g, one can obtain different solutions.

The minimal difference among the $x_{i+1}/V_{i+1} - x_i/V_i$, $i = \overline{1,n-1}$ ones is equal: to 15 if g = 1, to 74 if g = 2, to 175 if g = 3 and to 214 if g = 4. So, it is confirmed the fact that the larger the value of g, the larger the mentioned difference. Thus, if it is needed to increase this difference, one has to increase the value of g. But the value of g is limited from above by the value of $\lceil Q/n \rceil$ (approximately). In Examples 1-4, one has $Q = V/M = 20000/279 \approx 71.7$ and $\lceil Q/n \rceil = \lceil 71.7/20 \rceil = 4$. At the same time, the attempt to obtain the solution at g = 5, was unsuccessful.

6. Some properties of parameter *g*

As identified in Section 5, the upper limit of the g value depends on ΔM and may be on other factors. In Figure 2, the dependence on ΔM of the maximal value of g, $g_{\max}$, for which it was possible to obtain the solution according to the A_{HS} algorithm, is shown; initial data are the same as in Examples 1-4, except the values of g and ΔM. Parameter ΔM takes values in the interval [1; 19], where 19 = n – 1. The $g_{\max}$ value equal to 0 corresponds to cases where the solution was not obtained.

From Figure 2 one can see that the $g_{\max}$ value is small at small or large values of ΔM and is large – at medium values of ΔM in the interval [1; 19], with some mirror symmetry. To extend the possible properties of parameter g, were done respective calculations also for other two cases:

a) M = 227, n = 11, V = 20000 and the values of x_i, $i = \overline{1,11}$ equal to those of the first 11 beneficiaries in Example 1;

b) M = 127, n = 5, V = 20000 and the values of x_i, $i = \overline{1,5}$ equal to those of the first five beneficiaries in Example 1.

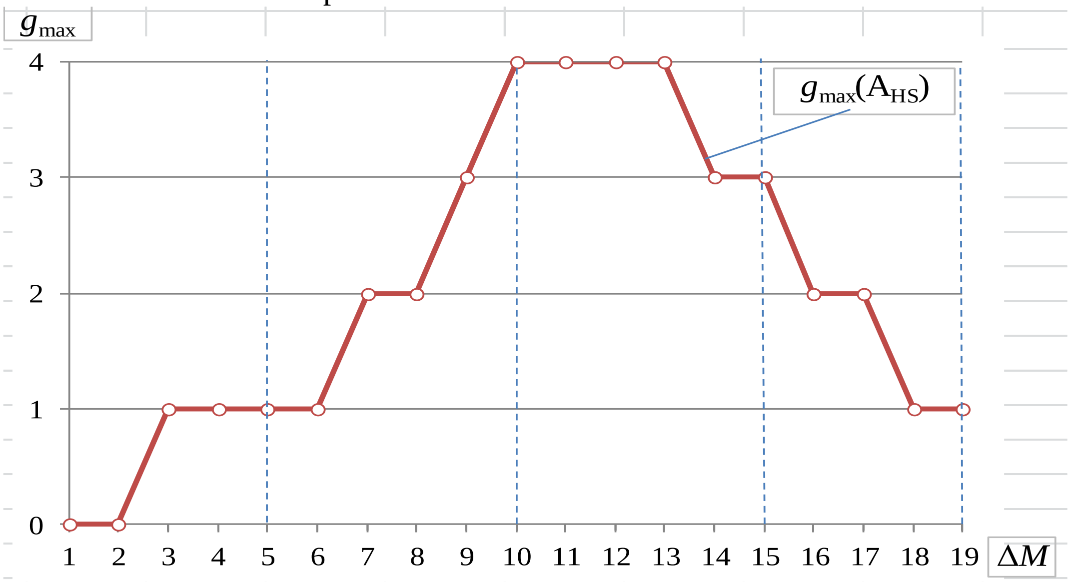

Figure 2. Dependence of $g_{max}(A_{HS})$ on ΔM.
Source: elaborated by the author.

The results for these two cases (Case (a) and Case (b)), obtained using SIMAP, are systemized in Table 5.

Table 5. Dependence of g_{max} on ΔM for Cases (a) and (b)

		ΔM									
		1	2	3	4	5	6	7	8	9	10
g_{max}	Case a	1	2	4	6	7	7	6	5	3	2
	Case b	14	30	21	12						

Source: elaborated by the author.

In **Case (a)**, one has $\lceil Q/n \rceil = \lceil 20000/(227\times11) \rceil \approx \lceil 8.0096 \rceil = 9$. The maximal value of $g_{max}(A_{HS})$ is 7, not reaching 9, but relatively close to it. The character of the dependence is similar to that in Figure 2, except the fact that for all $1 \le \Delta M \le 10$ it exist at list one solution ($g_{max} > 0$).

In **Case (b)**, one has $\lceil Q/n \rceil = \lceil 20000/(127\times5) \rceil \approx \lceil 31.5 \rceil = 32$. The maximal value of $g_{max}(A_{HS})$ is 30, not reaching 32, but relatively close to it. The character of the dependence is similar to that in Figure 2, except the fact that for all $1 \le \Delta M \le 4$ it exist many solutions: $12 \le g_{max}(A_{HS}) \le 30$.

Based on obtained data, with refer to parameter g one can conclude that:

1) the larger the value of g, the larger the minimal difference among the $x_{i+1}/V_{i+1} - x_i/V_i$, $i = \overline{1, n-1}$ ones;
2) the maximal value of g, g_{max}, for which it is possible to obtain the solution according to the A_{HS} algorithm, strongly depends on the value of ΔM and can vary from 0 to approximately $\lceil Q/n \rceil$;
3) at ones and the same initial data, the $g_{max}(A_{HS})$ value is small at small or large values of ΔM and is large – at medium values of ΔM in the interval $[1; n - 1]$, with some symmetry;
4) the approximation by $\lceil Q/n \rceil$ of the upper limit for the g_{max} value at $1 \le \Delta M \le n - 1$ is relatively good.

Finally, as was mentioned above, the use of parameter g aims to increase the value of the minimal difference among the $x_{i+1}/V_{i+1} - x_i/V_i$, $i = \overline{1, n-1}$ ones – for the apportionments that fully favor small beneficiaries, and (in another research using the A_{HL} algorithm) of the minimal difference among the $x_i/V_i - x_{i+1}/V_{i+1}$, $i = \overline{1, n-1}$ ones – for the apportionments that fully favor large beneficiaries, that is, in both cases, among the $\delta_i = |x_{i+1}/V_{i+1} - x_i/V_i|$, $i = \overline{1, n-1}$ ones. But sometimes it may be of interest to equalize these differences as much as possible, for example in order to minimize the value of the sum

$$\sum_{i=1}^{n-1} |\delta_i - \delta|\,, \delta = \frac{1}{n-1}\sum_{i=1}^{n-1} \delta_i. \tag{13}$$

Such a goal can be achieved by some modifications to 6 of the A_{HS} algorithm.

7. Conclusions

In order to determine Hamilton apportionments which fully favor beneficiaries, the A_{HS} algorithms was elaborated. It guarantees the solution (if it exists), regardless of the value of n. This algorithm was implemented in the computer application SIMAP. Four examples of calculations at $n = 20$ using SIMAP are described – there were generated four apportionments which fully favor small beneficiaries at different values of parameter g.

All four obtained apportionments fully favor small beneficiaries even if the n value is relatively large ($n = 20$). In this context, it should be noted that in all 25 million variants of initial data with $n = 20$, for which the V_i, $i = \overline{1, n}$ values were generated stochastically at uniform distribution, none of the Hamilton apportionments, obtained using SIMAP (Bolun, 2021), does not fully favor the beneficiaries.

At the same time, it was identified that the results of calculations depends considerably not only on the initial data V, n, $1 \leq \Delta M \leq n - 1$ and x_i, $i = \overline{1,n}$, but also on the parameter g value of the A_{HS} algorithm. It was identified that the higher the g value ($1 \leq g \leq \lceil Q/n \rceil$), the larger the minimal difference among the $x_{i+1}/V_{i+1} - x_i/V_i$, $i = \overline{1,n-1}$ ones. At the same time, the maximal value of g, g_{max}, for which it is possible to obtain the solution according to the A_{HS} algorithm, strongly depends on the value of ΔM, being small at small or large values of ΔM and large – at medium values of ΔM in the interval $[1; n - 1]$.

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