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ON BEHAVIOR OF GEODESICS ON A PAIR OF HYPERBOLIC PANTS

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Abstract: *We first discuss known rezults of geodesics on pair of pants and then propose new research problems about behavior of geodesics on a pair of hyperbolic pants (general, symmetric and generalized). This article presents a new way to classify geodesics on a pair of pants. We show that a geodesics on hyperbolic pair of pants can be classiffied into types.*
Keywords: *behavior of geodesics, the multilateral, the method of colour multilaterals, hyperbolic right angled hexagon, surfaces of signature (0,3)*

JEL CLASSIFICATION: MS; C 53; C60; F60; C22

1. Introduction

This paper is devoted to classify all geodesic curves on a generalized hyperbolic pair of pants by using the new constructive method - generalized coloured multilaterals method. In this paper we review some important results on the behavior of geodesics on pair of pants and discuss some extension of thouse results to the case on an arbitrary hyperbolic surface. Special thanks go to Professor V.S. Makarov who inspired me to do this work. The results ofer only a first glimpse on the behavior of geodesics on an arbitrary 2-dimensional hyperbolic manifold. One of the most important problems in geometry is to find the shortest path or geodesic between the two points. Behavior of geodesics in hyperbolic surfaces has been a fruitful subject of research for many years. The chaotic behavior of geodesics on surface of constant negative curvature and finite volume has been known since Hadamard (1898). The problem of understanding the geometry and dynamics of geodesics and rays (i.e. distance-minimizing half geodesics) on hyperbolic manifolds dates back at least to Artin, who started to study the qualitative behavior of geodesics on hyperbolic surfaces. Emil Artin studied the global behavior of geodesics on hyperbolic surfaces by cleverly encoding geodesics using continued fractions. A detailed analysis of the behavior of these geodesics is a problem of immense complexity. The behavior of geodesics on arbitrary hyperbolic surface is largely unexplored territory. The present work offers a few tools for further exploration. Based on the present analysis, we may adress issues like the global behavior of these geodesics. Furthemore the presented analysis offers tools for a systematic investigation of specific families of geodesics such as orthogeodesics, ideal geodesics. A geodesic in a hyperbolic surfaces is an arc which in the local coordinate charts, is the image of a geodesic arc of the hyperbolic plane. Topological surfaces are often thought of as the result of pasting togetheter polygons. Provided you have enough topology, pants decompositions are a natural way of decomposing (orientable) surfaces or conversely one can build a surface by pasting 3 holed spheres (pants) a long their cufs. So it is important to study the behavior of geodesics on a pair of pants.We start by considering this problem on hyperbolic pants in hopes of discovering a method which can be easily generalized to the problem of behavior of geodesic on any hyperbolic surface. We will explore how geodesics behave on a given pair of hyperbolic pants. The geodesics curves on pair of pants have been topics of interest in geometry. We also allow the degenerate case in which one or more of the lenths vanish. We call a generalized pair of pants a hyperbolic surface which is a homeomorphic to a hyperbolic sphere with three geometric holes, a hole being either a geodesic boundary component or a puncture whose neighborhood is a cusp. Symmetric hyperbolic pairs of pants, that is, hyperbolic

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pair of pants which have three geodesic boundary components of equal lenths. Geodesics are those curves on the surface that are not geodesically curved. Considering their role on surfaces they can be compared to straight lines in plane and they are called "the most straight lines" on surface. One of the main approaches to study of the geometry of manifolds and understand its structure is through the investigation of geodesics, the shortest path or geodesic between the two points. Behavior of geodesics in hyperbolic surfaces has been a fruitful subject of research for many years. The definitions of geodesic lines in various spaces depend on the particular structure (metric, line element, linear connection) on which the geometry of the particular space is based. In the geometry of spaces in which the metric is considered to be specified in advance, geodesic lines are defined as locally shortest. The local behavior of geodesic curves is similar to that of straight lines in Euclidean space. A sufficiently short arc of a geodesic line is the shortest among all rectifiable curves with the same ends. Only one geodesic line passes through any point in a given direction. Families of geodesic lines, considered as possible trajectories of motion, form a subject of the theory of dynamical systems and ergodic theory.

Thanks to the development of the new constructive approach, in this paper, the author succeeded to receive "in a certain sense" the solution for the behavior of the geodesics in general on the generalized hyperbolic pair of pants, structure of geodesics and their types. Geodesics solve the initial value problem which states that from any point of a manifold there starts a unique geodesic in any direction. In order to discuss the results of this work, it will be necessary to agree on some definitions of the basic concepts. The following terminology will be used regularly throughout this paper. A geodesic in a hyperbolic manifold is a locally distance - minimising curve, and is said to be simple if it has no transverse self-intersections (there-fore it is either an embedded copy of R or an embedded circle) and non simple otherwise. A geodesic on surface M is said to be complete if it is not strictly contained in any other geodesic, i.e., it is either closed and smooth, or open and of infinite length in both directions (its ends). Complete geodesics coincide with those which never intersect ∂M. Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp. Throughout, we use the term geodesic to refer to a complete infinite geodesic; a geodesic ray is a half-infinite ray; finally, a geodesic arc is a finite segment lying along some geodesic (which we assume to be closed unless otherwise stated). One says that a geodesic is complete if it is not strictly contained in any other geodesic. For a hyperbolic surface M some of the geodesics γ will come back to the point they start and fit in a smooth way. These are called closed geodesics. It ends up that there are finitely many closed geodesics of a given lenght (if any). Geodesics on smooth surfaces are the straightest and locally shortest curves. A hyperbolic surface is a surface which constant negative curvature. Unlike the plane, which is flat, or the sphere which has positive curvature, these hyperbolic surfaces are negatively curved. On a hyperbolic surface, some geodesics are infinitely long, like straight lines in the plane, but others close up into a loop, like the great circle on a sphere. Two basic properties are responsible for their importance: first, that from any point of a manifold there starts a unique geodesic in any direction. Second, the length minimization property (connecting two given points on a manifold with a locally shortest curves). On smooth surfaces geodesics possess both properties. Geodesics on hyperbolic surfaces are briefly discussed in (Balcan, 2021). Recall that the geodesics of the Poincare upper half-plane H, are the vertical (half-) lines and the semi-circles centered on the real line. A geodesic or any curve for that matter is simple if it contains no self intersections.

2. Behavior of geodesics on a given pair of hyperbolic pants (general, symmetric and generalized)

We want to describe their global behavior: a) when are geodesics closed? b) when are the dense? c) quantitatively, how do they wrap around the surface? These questions admit notably precise answers, as we are going to see. How do geodesics on the generalized hyperbolic pair of pants behave or how can we determine the behavior of a given geodesic on the hyperbolic pair of pants? The study of the geodesics on hyperbolic surfaces can be reduced to the study of the curves on a hyperbolic pair of pants. Let *P* be a pair of pants. We focus on getting the behavior of geodesics on a hyperbolic pair of pants *P*. Such geodesics are often studied by looking at their lifts in covering spaces of the surface. A hyperbolic pair of pants can be constructed by pasting alternate sades of two copies of a right-angled hyperbolic hexagon (along its three alternate sides). Yet another way to define a hyperbolic pair of pants is via its universal cover. There is a unique way to write *P* as the union of two congruent right-angled hexagons. Take this decomposition. Recall that two geodesics in a hyperbolic surface M are of the same type if their free homotopy classes differ by a homeomorphism of the surface. We classify the geodesics in types according to their geometric behavior. Now suppose we choose a point in the pair of pants and consider a spray of geodesics emanating from that pont. A geodesics is uniquely specified by a point and a tangent direction. Such geodesics are often studied by looking at their lifts in covering spaces of the pair of pants.

For the behaviour of the geodesics on the specified fragments (hyperbolic pants, etc.) it is used a certain figure, named in the text of the work *the multilateral.* The multilateral is obtained from a right angle hexagon as follows. We construct a hyperbolic hexagon with right angles on the hyperbolic plane H^2. For a certain value $r = r_1$ of the radius of the circle, inscribed in the right angle triangle, this triangle becomes limited: its vertices become infinitely remote points, and the sides - in pairs parallel lines. It is to be noted that these triangles decompose H^2. If we continue to increase the radius of the inscribed circle, then the sides of the triangle become pairwise divergent straight lines, the vertices - are ideal points and the area of the triangle is infinite. From the hyperbolic geometry we have that for given two disjoint geodesics on the plane H^2 with four different end points at the infinity (divergent), there is only one geodesic perpendicular to both. But, if from the obtained " beyond the limit" triangle we cut off the "excess" endless pieces with the help of common perpendiculars of the pairs of its sides, then we get an *equiangular-semi-regular hexagon*. All the angles of this hexagon are straight and the sides over one are equal. Symmetric right-angled hyperbolic hexagons, that is, convex right – angled hyperbolic hexagons having three non-adjacent edges of equal length. We'll call the newly appeared sides black, and „remnants" of the sides of the original triangle are white: than we can say that all the white sides are equal to each other in pairs, and all the black sides are also equal pairwise to each other, and the angle between the intersecting white and black sides is straight. Thus the resulting hexagon is a Coxeter, and the group generated by reflections in its sides is a Coxeter group. But before building all this Coxeterian partition, it is very useful to first make reflections only on the black sides of the hexagon and on their images, obtained by such reflections. Continuing indefinitely the reflection in these black sides, we get some new kind of regular polygon (it would be more accurate to say - *the multilateral*). The sides of this „ *multilateral*" (without vertices and angles) are straight lines, tangents (regular) system of circumferences on the hyperbolic plane . Obviously, the reflections in the sides of the straight lines of this multilateral can cover the whole (entire) hyperbolic plane H^2. To facilitate the understanding and further description, we agree to call the sides of the six-rectangle (right angled hexagon) black, if they are obtained from boundary geodesic circles of pants, and the other three sides we agree to consider painted in different colours (for example, red, blue and green straight). Exactly, this figure is also called in the work as a multilateral (in contrast to the polygon, the figure has no vertices and angles, hence its name - the multilateral). With the help of these multilaterals, it is possible to determine the nature of the behavior of the geodesics on the generalized hyperbolic pair of pants. A concrete method of investigating the behavior of the geodesics on the hyperbolic pants and their degenerations, in the subsequent consolidation of research results using the method proposed in this paper (sometimes called the method of generalized coloured multilaterals). The main purpose of this article is to indicate an algorithm (the construction of a practical approach) that allows determine the behavior of this geodesic on hyperbolic pair of pants. Also the aim is to obtain new results in following areas: a) the solution of the question of the qualitative behavior of the geodesics in general (if a point and the direction of the tangent at that point are given) on hyperbolic pair of pants; b) a new method for solving the problem of the behavior of the geodesics on hyperbolic pair of pants is developed - *the method of colour multilaterals*; c) with the help of this technique, the question of the qualitative behavior of the geodesics in general on hyperbolic pair of pants is solved. In more detail, the following main results of the study were obtained. A new constructive method for investigating the global behaviour of the geodesics on on the generalized hyperbolic pair of pants (the method of colour multilaterals) is given in this paper. Study of the behavior of the geodesics on the simplest hyperbolic surfaces (hyperbolic pants, degenerate hyperbolic pants, thrice-punctured sphere, etc.), some of which have long attracted the attention of geometers. In hyperbolic geometry, hyperbolic right angled hexagons are used as a tool for analysing the behaviour of the geodesic (and surfaces). Hyperbolic right angled hexagon; that is hexagon in the hyperbolic plane whose angles are all right angles. The discrete group Γ is defined in the usual way via its fundamental domain F (glued from the proper number of right angled hexagons). Hyperbolic pants are the only compact hyperbolic surfaces with a geodesic boundary that can't be simplified by cutting along closed simple geodesics. In fact, any pants with boundary geodesics are uniquely determined by the length of their boundary geodesics, because any hyperbolic right angled hexagon is uniquely defined by three alternating (non-adjacent) lengths of sides that can be arbitrarily set. We consider the universal covering of hyperbolic pants (the hyperbolic plane H^2) and lines that cover a given geodesic. Let it be on H^2 a right angled hexagon H and let H' denote its image under reflection from the side of δ_{13} (Fig. 1). When identifying the corresponding sides δ_{12} and δ_{12}, as well as δ_{23} and δ_{23} of this right angled geodesic octagon (Fig.1), we obtain hyperbolic pants *P* with boundaries α_1, α_2, α_3. As the fundamental region for the corresponding pants *P* we choose this hyperbolic right angled octagon (the double of a right angled hexagon H in the plane H^2). Two hyperbolic hexagons together forma an octagon which can be glued into a pair of pants.

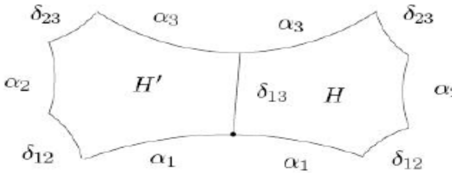


Fig. 1 The hexagon H and its reflection at the side δ_{13} is a symmetrical hexagon H'

This geodesic right angled octagon is the fundamental domain of the group $\Gamma = \langle t_1, t_2, t_3 \rangle$ generated by hyperbolic translations t_1, t_2, t_3, (the pants can be obtained by factorizing the hyperbolic plane H^2 by a discrete co-compact group Γ generated by translations t_1, t_2, t_3, where the translation t_i is determined by the vector $2 \cdot \alpha_i, i = 1, 2, 3$). To describe the behaviour of an arbitrarily given (some) geodesic on hyperbolic pants emanating from the point A in a given direction, we need to consider how the direct, covering this geodesic, behaves on the universal cover of these pants. In other words, how this straight line is located relative to the sides of α_3, α_1, α_2 of the hyperbolic right angled octagon (the so-called "colour" straight - blue, green, red). Walking along hyperbolic octagon, we can't cross the boundary components α_3, α_1, α_2 (" coloured circumferences "), but we can pass through the sides δ_{13}, δ_{12}, δ_{23} of a hyperbolic hexagon (the so-called "black" sides). Along with the coloured sides, the categories of coloured angles are built. A pair of "adjacent" colour angles uniquely determines the next colour angle with the help of colour (coloured "straight lines - blue, green, red) or with the help of geodesic sides $\alpha_1, \alpha_2, \alpha_3$. Suppose, for the beginning, point A is fixed on the surface of hyperbolic pants, and we need to understand how the geodesic's behaviour depends on the direction (from the directing vector emanating from point A). In this situation each side determines the angle of its colour with the vertex in the point A (and the sides parallel to the colour side) and in each category of angles it is uniquely determined which sides δ_{13}, δ_{12}, δ_{23} (or "black" sides) it is necessary to cross to be within the scope of the corresponding colour side. Thus, on hyperbolic pants, the problem of the behaviour of any geodesic passing through a fixed point is uniquely solvable by the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral obtained from a right angled hexagon. Thus, *on hyperbolic pants is the problem of the behaviour of any geodesic* that passes through a fixed point and is uniquely solvable with the help of *the algorithm for constructing the corresponding system of coloured angles, and by the sides parallel to the considered side of the generalized multilateral*, obtained from a right angled hexagon. Further, the concept of the category of angles is introduced, and with the help of these categories an algorithm for recognizing the type of a geodesic is given.

Every geodesic curve on the pair of pants is of one of the types. It is found special case: behavior of ortho-boundary geodesics and orthogeodesics, and their general structure, i.e., it is obtained classification of geodesics launched (emanating) normally from the point of geodesic boundary of pants. Is said to be orthogeodesic - a geodesic segment perpendicular to the boundary at its initial and terminal points (see Fig. 2). The construction can be extended to the situation when both remaining boundary components of the pair of pants are represented by cusps. There are studied geodesics on generalized hyperbolic pants (a sphere with *b* boundary components and *p* cusps, with $b+p=3$) and on hyperbolic thrice punctured sphere. It is proved that in two dimension the only such manifold not containing a simple closed geodesic is the hyperbolic thrice punctured sphere. But it has six simple complete geodesics.

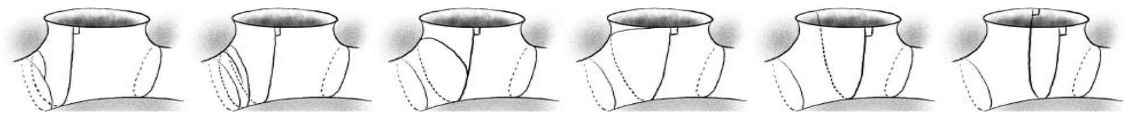
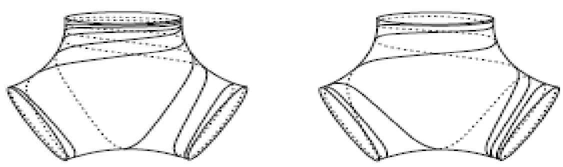


Fig. 2. **A geodesic segment perpendicular to the boundary at its initial and terminal points**

Only 21 geodesics are simple-namely 3 closed geodesics (the boundary components, 6 geodesics spiraling from a boundary component to itself, and 12 geodesics spiraling from a boundary component to another. They differ by the spiraling directions at each boundary component (see Fig. 3).



References:

Balcan, V. (2021) A new way to classify geodesics on hyperbolic manifolds. *Mathematics and IT:Research and Education 2021, Virtual Internat. Conf., Moldova*, 12-13.